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GEOMETRY OF
EINSTEIN'S UNIFIED FIELD THEORY

TO THE MEMORY OF A GENIUS

ALBERT EINSTEIN

1879–1955

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GEOMETRY OF EINSTEIN'S UNIFIED FIELD THEORY

P. NOORDHOFF LTD. — GRONINGEN — HOLLAND

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Bloomington, Indiana
Autumn, 1957

Václav Hlavatý

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PREFACE

I. GENERAL OUTLINE

In Appendix II to his last book (6) [1] Einstein proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. It may be characterized as a *set of geometrical postulates for the space-time* X_4. The geometrical consequences of these postulates are not developed very far by Einstein.

Successful physical application of this theory will be engendered by a complete knowledge of the geometrical structure of the space-time implied by the postulates. *The main purpose of this book is to provide a detailed geometrical background for physical applications of the theory.* It so happens that the detailed investigation of Einstein's geometrical postulates opens an easy way to a physical interpretation. Even this physical interpretation is based on geometry rather than on physics.

The gravitational field is described by ten components of a symmetric quadratic tensor, while the electromagnetic field is defined by a quadratic skew symmetric tensor described by six components. Therefore, we may expect that a unified theory comprising gravitation and electromagnetism be based on sixteen field quantities. Several attempts in this direction were made by Einstein and ultimately discarded. [2] His last attempt, which he regarded as final, is described in this book.

The geometrical postulates governing our theory may be described in terms of three principles.

Principle A. The unified field of gravitation and electromagnetism is determined by sixteen potentials.

Principle B. These potentials determine the curvature and the torsion of the four-dimensional space-time.

---

[1] Numbers in parentheses refer to the bibliography at the end of the book.

[2] In the literature there are many approaches to the problem of the unified field theory. Some of them strongly influenced the development of geometry, although none has received general recognition as a physical theory.

Principle C. The set of sixteen potentials is a solution of a system of differential field equations imposing certain conditions on the curvature and torsion of space-time. The identification of gravitational and electromagnetic fields with the potentials has to be derived from the field equations.

A more detailed description of the three principles is exhibited in the following:

Principle A concerns the algebraic structure imposed on space-time X_4 by a *general* real tensor $g_{\lambda\mu}$. This tensor has sixteen components and may be split into its symmetric part $h_{\lambda\mu}$ and its skew symmetric part $k_{\lambda\mu}$ [1]:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}. \tag{I.1}$$

The tensor $h_{\lambda\mu}$ is assumed to possess the signature $(+ + + -)$. Its determinant will be denoted by $\mathfrak{h}$. Its contravariant components will be denoted by $h^{\lambda\nu}$. The tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ are used to lower and raise indices. In particular, we put

$$k_{\mu}{}^{\nu} \overset{\text{def}}{=} k_{\mu\lambda}h^{\lambda\nu}$$

and consider the roots of the following equation

$$\operatorname{Det}\left(\left(k_{\mu}{}^{\nu} - \lambda\delta_{\mu}^{\nu}\right)\right) = 0. \tag{I.2}$$

We say that space-time (or either one of the tensors $g_{\lambda\mu}$, $k_{\lambda\mu}$) is of *the first class*, if all four roots $\underset{1}{\lambda}, \ldots, \underset{4}{\lambda}$ are different from zero,

the second class, if only two of the roots are different from zero,

the third class, if all roots are equal to zero.

In the first class there are exactly four linearly independent vectors which satisfy the equation

$$\lambda a^{\nu} = k_{\mu}{}^{\nu}a^{\mu}\,.^{2} \tag{I.3}$$

Each of these vectors is also a null vector

$$a^{\lambda}a^{\mu}h_{\lambda\mu} = 0. \tag{I.4}$$

In the second class we have two linearly independent eigenvectors satisfying (I.3) for $\lambda \neq 0$ and (I.4). Moreover, there is a one-parameter set of vectors satisfying (I.3) for $\lambda = 0$. Among these there are only two vectors satisfying (I.4). Thus, in each of these classes we have four linearly independent vectors $\underset{1}{a}^{\nu}, \ldots, \underset{4}{a}^{\nu}$ satisfying (I.3) and (I.4). In the third class there is a one-parameter set of vectors satisfying (I.3) for $\lambda = 0$. Among these vectors there is only one, say $\underset{4}{a}^{\nu}$, satisfying also (I.4). We may choose arbitrarily three vectors $\underset{1}{a}^{\nu}$, $\underset{2}{a}^{\nu}$, $\underset{3}{a}^{\nu}$ in such a way that $\underset{1}{a}^{\nu}, \ldots, \underset{4}{a}^{\nu}$ are linearly independent. The minimum number of arbitrary parameters involved in this choice is two. It should be noted that these parameters are additional parameters not connected with the given tensor $g_{\lambda\mu}$.

---

[1] The reader not familiar with the notation of tensor calculus will find a brief introduction to it in the third part of the Preface.

[2] A vector satisfying (I.3) is often called an eigenvector of $k_{\mu}{}^{\nu}$.

The vectors $\underset{1}{a}^{\nu}, \ldots, \underset{4}{a}^{\nu}$ will be termed *basic vectors.*

Further discussion is easier in terms of projective geometry. Denote by T_4 the tangent space of X_4 at a generic point O and by P_3 the ideal (projective) space of T_4. The introduction of P_3 into our discussion is justified by the fact that the transformation

$$'h_{\lambda\mu} = \alpha h_{\lambda\mu}, \quad 'k_{\lambda\mu} = \beta k_{\lambda\mu}$$

does not change the classes. Hence, the above classification depends only on the ratios of the components $h_{\lambda\mu}$ and on the ratios of the components $k_{\lambda\mu}$. If we consider only these ratios, then we are led in a natural way to the projective geometry in P_3. The tensor $h_{\lambda\mu}$ determines a quadric H in P_3 by the ratios of its components, while the ratios of the components $k_{\lambda\mu}$ determine a linear line complex K in P_3. In other words, the components $k_{\lambda\mu}$ are to be thought of as Plücker's projective homogeneous plane-coordinates of K in P_3.[1] This complex is a *general* one for only the first class. Then there are exactly two skew lines L and L^* which are conjugate polars with respect to H and K. They intersect H in four different points $\underset{1}{a}^{\nu}, \ldots, \underset{4}{a}^{\nu}$ which are the ideal points of the basic vectors $\underset{1}{a}^{\nu}, \ldots, \underset{4}{a}^{\nu}$ of $k_{\mu}{}^{\nu}$ in T_4. The complex is a *special* one for the last two classes, and its axis is the line L of Plücker coordinates $k_{\lambda\mu}$. In the case of the second class it intersects H in two different points, so that its conjugate line L^* with respect to H intersects this quadric in another pair of distinct points. Again we obtain in this way four different points $\underset{1}{a}^{\nu}, \ldots, \underset{4}{a}^{\nu}$ on H, which are the ideal points of four basic vectors of $k_{\mu}{}^{\nu}$. For the third class, both lines L, L^* meet in the point $\underset{4}{a}^{\nu}$ which is the common point of contact of L, H and L^*, H. It is the ideal point of the eigenvector $\underset{4}{a}^{\nu}$ of $k_{\mu}{}^{\nu}$ in T_4, this eigenvector being the only eigenvector that is a null vector.

---

[1] A brief introduction to line geometry is given in Appendix III

Therefore, we see that unlike the gravitational theory[1] the unified theory exhibits *preferred points* on H, points which are obtained from the field quantities $g_{\lambda\mu}$. For the first two classes there are four such points, while for the third class there is only one. However, in this latter case it is possible to find another set of three points, $\underset{1}{a}^{\nu}, \ldots, \underset{3}{a}^{\nu}$, to form with $\underset{4}{a}^{\nu}$ a set of four linearly independent points. Hence, in all three cases we have a set of four linearly independent points, $\underset{1}{a}^{\nu}, \ldots, \underset{4}{a}^{\nu}$, but only for the first two classes is their position completely defined by the given data.[2]

Note. Although the electromagnetic field $f_{\lambda\mu}$ is always different from $k_{\lambda\mu}$, it belongs to the same class as does $k_{\lambda\mu}$. Therefore, there are three different classes of electromagnetic fields. They may be characterized by their Lorentz invariants. Thus, for instance, the plane wave in the electromagnetic theory of light is of the third class, while the electromagnetic field of a moving electron is of the second class, and still more complicated electromagnetic fields belong to the first class.

The presence of $k_{\lambda\mu}$ makes it impossible to apply the perturbation method customary in the gravitational theory. For example, there one can make the expansion

$$h_{\lambda\mu} = \overset{0}{h}_{\lambda\mu} + \epsilon \overset{1}{h}_{\lambda\mu} + \ldots$$

where $\overset{0}{h}_{\lambda\mu}$ is a set of constants of the signature $(+\,+\,+\,-)$, and ϵ is a small constant so that in the results corresponding to this approximation we may neglect all terms multiplied by ϵ^q ($q = 2$, 3, ...). This approximation *does not change the structure of H*. In the unified theory a similar approximation must at least be of the form

---

[1] By gravitational theory we mean the theory usually called the general theory of relativity. This theory is based on a symmetric tensor, say $h_{\lambda\mu}$.

[2] The components of the corresponding vectors $\underset{1}{a}^{\nu}, \ldots, \underset{4}{a}^{\nu}$ are defined *in all three cases* up to a two-parameter group of linear transformations. They constitute the nonholonomic frame of reference which is very important for Principle B.

$$g_{\lambda\mu} = \overset{0}{h}_{\lambda\mu} + \epsilon \overset{1}{g}_{\lambda\mu} + \cdots.$$

If one neglects the terms multiplied by ϵ^q, one excludes a priori the first two classes, for $q = 2, 3, \ldots$. Therefore, this kind of approximation is limited to the third class only. [1]

Note. The tensor $g_{\lambda\mu}$ is the most natural tool for the four-component spinor algebra and analysis (Appendix II).

Principle B. The tensor $g_{\lambda\mu}$ imposes a differential-geometric structure on space-time X_4 through the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ defined by the following system of sixty-four equations

$$\text{(I.5)} \qquad \partial_\omega g_{\lambda\mu} - \Gamma_{\lambda}{}^{\alpha}{}_{\omega} g_{\alpha\mu} - \Gamma_{\omega}{}^{\alpha}{}_{\mu} g_{\lambda\alpha} = 0, \quad \left(\partial_\omega \overset{\text{def}}{=} \frac{\partial}{\partial x^\omega}\right).$$

This system of sixty-four equations is similar in structure to the system of forty equations which define the Christoffel symbols in the gravitational theory. The only striking difference between these two systems is the position of the indices $\omega\mu$ in $\Gamma_{\omega}{}^{\alpha}{}_{\mu}$. [2]

However, this peculiar order of the indices $\omega\mu$ has a profound geometrical meaning. If we put $\Gamma_{\mu}{}^{\alpha}{}_{\omega}$ instead of $\Gamma_{\omega}{}^{\alpha}{}_{\mu}$ in (I.5), the resulting system admits a solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ only if $g = \text{const.}$, where $g = Det((g_{\lambda\mu}))/\mathfrak{h}$. We shall see in Chapter V that this scalar is closely connected with the gravitational function. Therefore, the requirement $g = \text{const.}$ would be a strong restriction of generality which we cannot afford at this stage of development. In order to explain Einstein's physical interpretation of this peculiar order, we define the tensor $\bar{g}_{\lambda\mu}$ by

$$\bar{g}_{\lambda\mu} \overset{\text{def}}{=} h_{\lambda\mu} - k_{\lambda\mu} = h_{\mu\lambda} + k_{\mu\lambda} = g_{\mu\lambda}$$

and denote by $\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$ the same function of $\bar{g}_{\alpha\beta}$ as $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is of $g_{\alpha\beta}$. If one *assumes that* (I.5) *admits exactly one solution* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, then one proves easily that $\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{\mu}{}^{\nu}{}_{\lambda}$.[1] Therefore, the system (I.5) holds even if we replace $g_{\alpha\beta}$, $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ with $\bar{g}_{\alpha\beta}$, $\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$. In (7), p. 144, Einstein interprets this situation as follows: "This fact is the mathematical expression of the indifference of the theory to the sign of electricity." This is an intuition of genius, since Einstein was not in possession of the tensor $f_{\lambda\mu}$ of electromagnetism.

---

[1] In fact one has to start with $q > 4$ for the first class and with $q > 2$ for the second class if one wants to preserve the class in this kind of approximation.

[2] The position of these indices is irrelevant in the gravitational theory because the Christoffel symbols are symmetric in the lower indices.

The system (I.5) does not always admit a solution. Necessary and sufficient conditions for the existence and uniqueness of a solution of (I.5) are $g \neq 0$ for the first class and $g(g - 2) \neq 0$ for the second class. In the case of the third class the system (I.5) always admits a unique solution.

Although the solution of (I.5) could be found, at least in principle, by means of elementary algebra, the result would not have the tensorial form we shall need in considering Principle C. It is clear that only a solution obtained by *tensorial operations* will lead to such a form. Therefore, we shall use only tensorial operations to derive this solution. First we rewrite (I.5) in tensorial form

$$D_{\omega} g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}. \tag{I.6}$$

Here D_{ω} is the symbolic vector of the covariant derivative with respect to $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, and

$$S_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=} \Gamma_{[\lambda}{}^{\nu}{}_{\mu]}.\text{[2]} \tag{I.7}$$

The equation (I.6) can be split into two

$$D_{\omega} h_{\lambda\mu} = 2S_{\omega(\mu}{}^{\alpha} g_{\lambda)\alpha} \tag{I.8)a}$$

$$D_{\omega} k_{\lambda\mu} = 2S_{\omega[\mu}{}^{\alpha} g_{\lambda]\alpha}. \tag{I.8)b}$$

A procedure similar to Christoffel's elimination applied to (I.8)a yields

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu} \tag{I.9}$$

where

$$U^{\nu}{}_{\lambda\mu} \overset{\text{def}}{=} 2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta}, \tag{I.10}$$

and $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$ are Christoffel's symbols of $h_{\lambda\mu}$. From the point of view of the actual solution $\Gamma_\lambda{}^\nu{}_\mu$ of (I.5), the equation (I.9) does not tell us very much, for the unknowns $\Gamma_\lambda{}^\nu{}_\mu$ appear on both sides of it. On the other hand, we see from (I.9) that if (I.5) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, this solution must be of the form indicated by (I.9). Therefore, the problem of solving (I.5) reduces to finding the tensor $S_{\lambda\mu}{}^\nu$ in terms of $g_{\lambda\mu}$. If we substitute into (I.8)b from (I.9), we obtain, after some rearrangement,

---

[1] Einstein assumed this relation.

[2] For the operations [ ] and ( ), *cf.* the third part of the Preface.

$$\text{(I.11)a} \qquad 2S_{\alpha\beta\gamma}X^{\alpha\beta\gamma}_{\omega\mu\nu} = K_{\omega\mu\nu},$$

where

$$\text{(I.11)b} \qquad X^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\text{def}}{=} \delta^{\alpha\beta\gamma}_{[\omega\mu]\nu} - 2\delta^{[\alpha}_{\nu} k_{[\omega}{}^{\beta]} k_{\mu]}{}^{\gamma} - 2\delta^{[\alpha}_{[\mu} k_{\omega]}{}^{\beta]} k_{\nu}{}^{\gamma},$$

$$\text{(I.11)c} \qquad K_{\omega\mu\nu} \overset{\text{def}}{=} \nabla_\omega g_{\nu\mu} + \nabla_\mu g_{\omega\nu} + \nabla_\nu g_{\omega\mu},$$

and ∇_ω is the symbolic vector of the covariant derivative with respect to $\{\lambda{}^\nu{}_\mu\}$. In order to solve the system (I.11)a of twenty-four equations for twenty-four unknowns $S_{\alpha\beta\gamma}$, we first restrict ourselves to the first two classes. If we use the basic vectors $\underset{i}{a}^\nu$ as a non-holonomic coordinate frame, and rewrite (I.11)a in this frame, we find that the twenty-four equations (I.11)a reduce to essentially three equations. Solving these and returning to the general coordinate system, we obtain

$$\text{(I.12)} \qquad 2S_{\omega\mu\nu} = (Z^{\alpha\beta\gamma}_{\omega\mu\nu} + 3k_\omega{}^\xi k_\mu{}^\eta Z^{\alpha\beta\gamma}_{[\xi\eta\nu]})K_{\alpha\beta\gamma}.$$

Here $Z^{\alpha\beta\gamma}_{\omega\mu\nu}$ is an explicitly defined tensorial function of $g_{\lambda\mu}$. This solution exists if and only if $g \neq 0$ for the first class and $g(g-2) \neq 0$ for the second class.

Although a similar procedure could also be applied to the third class, another, more direct method leads in this case to the solution of (I.11)a in the form

$$\text{(I.13)} \qquad 2S_{\omega\mu\nu} = K_{\omega\mu\nu} - 2k_{[\mu}{}^\alpha K_{\omega]\alpha\beta} k_\nu{}^\beta.$$

Substituting from (I.12) ((I.13)) into (I.10) and (I.9), one obtains the solution of (I.5) in the form (I.9) expressed explicitly in terms of $g_{\lambda\mu}$ and $\nabla_\omega g_{\lambda\mu}$

$$\text{(I.14)} \qquad \Gamma_\lambda{}^\nu{}_\mu = \Gamma_\lambda{}^\nu{}_\mu(g_{..}, \nabla_{.} g_{..}).$$

Note I. The third class is characterized by the conditions

$\underset{1}{\lambda} = \dots = \underset{4}{\lambda} = 0$. If we substitute these values into (I.12),[1] we obtain an equation which is different from (I.13) unless some very strong geometrical requirements about the vectors $\underset{j}{a}^\nu$ are fulfilled. Hence, from the point of view of Principle B, *there are two unified theories, one for the first two classes and one for the third class.*

Note II. The question arises whether the system (I.5) admits at least one solution, if $g(g-2) = 0$. To answer this question we mention here only the case $g = 2$: A necessary and sufficient condition that (I.5) admit at least one solution for $g = 2$ is that the vectors $\underset{j}{a}^\nu$, $j = 1, 2$ be tangent vectors to geodesic lines. Then (I.5) admits more than one solution. It is not difficult to find their explicit form. A different requirement about the vectors $\underset{1}{a}^\nu, \dots, \underset{4}{a}^\nu$ holds for the case $g = 0$.

Principle C. After having found the solution (I.14) of (I.5) we substitute it into the equations defining the curvature tensor $R_{\omega\mu\lambda}{}^\nu$ of $\Gamma_\lambda{}^\nu{}_\mu$. In this way the curvature tensor appears as a function of $g_{\lambda\mu}$ and its first two derivatives, and the same holds for the contracted curvature tensor $R_{\mu\lambda} \overset{\text{def}}{=} R_{\alpha\mu\lambda}{}^\alpha$: In order to obtain the tensor $g_{\lambda\mu}$ with which we started in dealing with Principles A and B, we may prescribe some conditions for it in terms of $R_{\mu\lambda}$ and $S_{\lambda\mu}{}^\nu$. Einstein suggested several different sets of such conditions. His final suggestion consists of three sets of differential tensorial equations

(I.15)a $$S_\alpha = 0 \quad (S_\alpha \overset{\text{def}}{=} S_{\alpha\lambda}{}^\lambda)$$

(I.15)b $$E^\nu = 0$$

(I.15)c $$R_{(\mu\lambda)} = 0.$$

Here

(I.16)a $$\sqrt{|\mathfrak{h}|}E^\nu \overset{\text{def}}{=} \mathfrak{E}^{\omega\mu\lambda\nu}\partial_\omega R_{\mu\lambda},$$

and $\mathfrak{E}^{\omega\mu\lambda\nu}$ is the contravariant indicator with the components $+1$, -1, 0. Later on we shall be compelled to replace (I.16)a with

(I.16)b $$E^\nu \overset{\text{def}}{=} D_\lambda GQ^{\lambda\nu} - 2U^\nu{}_{\lambda\mu}H^{\lambda\mu} - U^\beta{}_{\beta\alpha}(GQ^{\alpha\nu} + 2Hh^{\alpha\nu})$$

where $GQ^{\lambda\nu}$ is a tensor to be discussed later, $H_{\lambda\mu} \overset{\text{def}}{=} H_{\alpha\lambda\mu}{}^\alpha$ is the contracted curvature tensor of $h_{\lambda\mu}$, and $H \overset{\text{def}}{=} h^{\lambda\mu}H_{\lambda\mu}$. For the time being we shall take for E^ν *either one* of the vectors defined by (I.16).

---

[1] and redefine suitably the tensor Z

We see that in the unified theory *the conditions* (I.15) *are of a purely geometrical nature and physical interpretation is not involved in them a priori.* Therefore, the program of study of Principle C may be stated as follows:

(1) an investigation of the structure of the curvature and torsion imposed on the space-time by aforementioned conditions,

(2) an attempt to identify the gravitational field and the electromagnetic field *by means of these conditions*, and

(3) an investigation of the physical consequences of (1) and (2).

The geometrical consequences of the conditions (I.15) will be discussed later in this Preface in a more detailed description of the fourth chapter. Therefore, we shall limit ourselves here to an outline of the basic ideas which govern problems (2) and (3).

Let us start with problem (2) and denote by $f_{\lambda\mu}$ a tensor field which could be identified with that of the electromagnetic field. Such a field must satisfy the Maxwell equations. Moreover, these equations must appear as a consequence of the purely geometrical conditions (I.15), since the Maxwell equations must constitute a part of the unified theory. We assume the Maxwell equations in their "idealized" form [1]

$$\text{(I.17)a}_1 \qquad \partial_{[\omega}f_{\mu\lambda]} = 0$$

$$\text{(I.17)b} \qquad v^{\nu}\sqrt{|\mathfrak{h}|} = \partial_{\lambda}f^{\lambda\nu}\sqrt{|\mathfrak{h}|}.$$

The equations (I.15) and (I.17) do not suffice for a unique identification of the electromagnetic field. Therefore, we add further restrictions. First, we require that $f_{\lambda\mu}$ be a function of $g_{\lambda\mu}$ but not of its derivatives. Second, we require that $f_{\lambda\mu}$ be obtainable from $g_{\lambda\mu}$ by a purely tensorial construction. This seems to be reasonable, since $f_{\lambda\mu}$ is itself a tensor. Of course, neither condition appears to be necessary. Only experiment can decide which requirements are appropriate. In any case, there is *only one* skew symmetric tensor $f_{\lambda\mu}$ (up to a multiplicative constant) having the following properties:

a) It satisfies both the above conditions.

b) If (I.15) holds, then (I.17)a$_1$ holds.

As a matter of fact it turns out that the equations (I.15)a and (I.17)a$_1$ are equivalent. Hence, the condition (I.15)a reduces the number of unknowns, for according to (I.17)a$_1$ the six unknowns $f_{\lambda\mu}$ are expressible in terms of four unknowns f_μ:

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[1] *i.e.*, we choose suitable units and take the dielectric constant and the magnetic permeability to be unity.

(I.17)a$_2$
$$f_{\lambda\mu} = \partial_{[\lambda} f_{\mu]}.$$

The tensor $f_{\lambda\mu}$ is given by

(I.18)a
$$f_{\lambda\nu} \overset{\text{def}}{=} \frac{\sigma}{\sqrt{g}} [\kappa\sqrt{k}\, k_{\lambda\nu} - \tfrac{1}{2} \mathfrak{e}_{\lambda\nu\alpha\beta} \sqrt{|\mathfrak{h}|}\, k^{\alpha\beta}].$$

Here σ and κ are well-defined signs, $k \overset{\text{def}}{=} Det((k_{\lambda\mu}))/\mathfrak{h}$ and $\mathfrak{e}_{\lambda\nu\alpha\beta}$ is the covariant indicator. If $k + 1 \neq 0$, then (I.18)a admits a unique solution

(I.18)b
$$k_{\lambda\nu} = \frac{\sigma}{\sqrt{e}} [\tfrac{1}{2} \mathfrak{e}_{\lambda\nu\alpha\beta} \sqrt{|\mathfrak{h}|}\, f^{\alpha\beta} - \varphi\sqrt{|f|}\, f_{\lambda\nu}]$$

where φ is a well-defined sign and e is an explicitly-defined scalar function of $f_{\lambda\mu}$ which may be expressed also as a function of $g_{\lambda\mu}$.

In order to prove that (I.17)b is a consequence of (I.15), we first *define* the vector v^ν by

(I.19)
$$\sqrt{|\mathfrak{h}|}\, v^\nu \overset{\text{def}}{=} \partial_\lambda f^{\lambda\nu} \sqrt{|\mathfrak{h}|}.$$

It is to be stressed that (I.19) is *a definition* of the vector v^ν while (I.17)b is *a set of differential equations* for the unknowns $f_{\lambda\mu}$. In order to convert (I.19) into (I.17)b we need, besides (I.19), *another determination* of v^ν in terms of $f_{\lambda\mu}$. Substituting from (I.18) and (I.19) into (I.14), we obtain $\Gamma_\lambda{}^\nu{}_\mu$ expressed in terms of v^ν, of $h_{\lambda\mu}$ and its first derivatives, and of $f_{\lambda\mu}$ and its first derivatives:

(I.20)
$$\Gamma_\lambda{}^\nu{}_\mu = \Gamma_\lambda{}^\nu{}_\mu(h_{..}, \partial.h_{..}, f_{..}, \nabla.f_{..}, v^\cdot).$$

Therefore, (I.15)b is of the form

(I.21)
$$E^\nu(h_{..}, f_{..}, v^\cdot, \nabla.v^\cdot, \nabla.\nabla.v^\cdot) = 0 \text{ [1]}.$$

Denote by

(I.22)
$$v^\nu = v^\nu(h_{..}, f_{..}) \text{ [1]}$$

the solution of the set of four differential equations (I.21) for four unknowns v^ν. (I.22) *is the other determination of* v^ν, *for which we were*

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[1] We are neglecting here the derivatives of the arguments $h_{\lambda\mu}$, $f_{\lambda\mu}$ because they are irrelevant for this type of global description of our problem.

looking. Substituting it into (I.19), we obtain four differential equations (I.17)b for four unknowns f_λ, involved in (I.17)a$_2$. In this way *we obtained Maxwell's equations* (I.17) *as a consequence of* (I.15)a,b. *This justifies the identification of the electromagnetic tensor field with the tensor field* (I.18)a. It is clear that $f_{\lambda\mu}$ depends on $h_{\lambda\mu}$

$$f_{\lambda\mu} = f_{\lambda\mu}(h_{..}).\ ^1 \tag{I.23}$$

In order to identify the gravitational field tensor we use the remaining restriction (I.15)c imposed on the curvature tensor. Upon substituting from (I.20) into (I.15)c, one obtains a system of ten equations

$$H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = \frac{G}{4}T_{\lambda\mu}. \tag{I.24}$$

Here the left hand term has the same meaning as in the field equations of the gravitational theory. $H_{\lambda\mu} \overset{\text{def}}{=} H_{\alpha\lambda\mu}{}^{\alpha}$ is the contracted curvature tensor of $h_{\lambda\mu}$, and $H \overset{\text{def}}{=} h^{\lambda\mu}H_{\lambda\mu}$. The scalar G is an explicitly given function of $f_{\lambda\mu}$, and the tensor $T_{\lambda\mu}$ has the following form

$$T_{\lambda\mu} = v_\lambda v_\mu + Q_{\lambda\mu}.$$

$Q_{\lambda\mu}$ is an explicitly given function of the arguments $f_{\lambda\mu}$, $\nabla_\nu f_{\lambda\mu}$ and v^ν, $\nabla_\lambda v^\nu$. This tensor is involved in the equation (I.16)b. The form of (I.24) is not affected by our choice of (I.16)a or (I.16)b. Comparing (I.24) with the field equations of the gravitational theory, we are encouraged *to identify the gravitational tensor field with* $h_{\lambda\mu}$, *the tensor* $T_{\lambda\mu}$ *with the momentum energy tensor for matter and electromagnetism, and* G *with the gravitational function.* [2] *In this way we have given physical significance to all the purely geometrical conditions* (I.15). From (I.23) and (I.24) we see that the *gravitational and electromagnetic fields are interrelated.* In the unified theory we have $g_{\lambda\mu} \neq g_{(\lambda\mu)}$, and therefore, the electromagnetic field cannot vanish. Therefore, the electromagnetic field may be looked upon as the primary field, and we may say that according to (I.24) *the gravitational field is generated by the electromagnetic field.* However, there are particular electromagnetic fields which do not generate gravi-

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[1] See footnote, page XIX.

[2] There are cases where G reduces to a constant.

tation. An example of such a field is the electromagnetic field of a plane wave in the electromagnetic theory of light.

One of the consequences of (I.24) is the conservation law

$$\nabla_\lambda G T^{\lambda\nu} = 0 \tag{I.25}$$

equivalent to the equations of motion,

$$v^\lambda \nabla_\lambda G v^\nu + \nabla_\lambda G Q^{\lambda\nu} = 0. \tag{I.26}$$

We obtain (I.26) regardless of whether we choose (I.16)a or (I.16)b. Assume first the validity of (I.16)a. Then the equations (I.26) are disappointing in many respects. They do not display the role of the unified connection $\Gamma_\lambda{}^\nu{}_\mu$. They introduce, besides the unified field

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}(f\cdot\cdot) \tag{I.27}$$

a kind of generalized Lorentz electromagnetic force vector $\nabla_\lambda G Q^{\lambda\nu}$.[1] Finally, they lead in celestial mechanics to predictions which are at variance with the predictions of the three crucial phenomena made by the gravitational theory.[2]

Assume now that (I.16)b holds, *i.e.*, assume

$$D_\lambda G Q^{\lambda\nu} = 2U^\nu{}_{\lambda\mu} H^{\lambda\mu} + U^\beta{}_{\beta\alpha}(GQ^{\alpha\nu} + 2Hh^{\alpha\nu}). \tag{I.28}$$

Then (I.26) is equivalent to

$$\frac{d^2x^\nu}{dp^2} + \Gamma_\lambda{}^\nu{}_\mu \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} = 0 \tag{I.29}$$

with a quadratic first integral

$$h_{\lambda\mu} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} = \text{const.} \tag{I.30}$$

Hence, we obtain the *unified law of inertia*: *If a particle moves freely in the unified field* (I.27), *it describes an autoparallel line of the unified connection* (I.9). If we assume the unified field $g_{\lambda\mu}$ to be of the form

$$g_{\lambda\mu} = h_{\lambda\mu}(O) + x^\alpha(\partial_\alpha g_{\lambda\mu})_O + \dots,$$[3]

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[1] We remind the reader that in the gravitational theory the concept of the gravitational force vector is replaced by the concept of the gravitational quadratic tensor field.

[2] *i.e.*, the advance of the perihelion of Mercury, the deflection of light and the gravitational shift of spectral lines.

[3] We may limit ourselves to the third class. The series under consideration cannot hold for *every* point if we assume $k_{\lambda\mu} \neq 0$.

then there is a coordinate system with the following properties:

a) At O three of the four equations (I.29) reduce to the classical Newtonian gravitational law, and the fourth equation reduces to an identity.

b) In the linear neighborhood of O the system (I.29) reduces to the gravitational relativistic second Newtonian law with the gravitational relativistic Lorentz electromagnetic force vector.[1]

In this way the concepts of the gravitational force vector and of the Lorentz electromagnetic force vector are replaced in the unified theory by the concept of the unified field $g_{\lambda\mu}$. Moreover, from the unified theory one obtains in the first approximation the classical Newtonian theory, and in the second approximation the gravitational theory with electromagnetic field included.

In applying this theory to celestial mechanics, we assume that the electromagnetic field of the sun is weak, so that products of the order of magnitude of $(f_{\lambda\mu})^q$, $q \geq 2$ may be neglected. Then (I.28) is identically satisfied, (I.24) reduces to

$$H_{\lambda\mu} = 0, \tag{I.31}$$

and (I.29) reduces to

$$\frac{d^2x^\nu}{dp^2} + \{\lambda{}^\nu{}_\mu\} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} = 0. \tag{I.32}$$

(I.31) and (I.32) are the equations of the gravitational theory which yield the above mentioned predictions. *Therefore, these predictions, obtained by means of the unified theory, are the same as in the gravitational theory.*

Having disposed in this way of the objections mentioned earlier, we have to add one more remark: We obtained the above predictions by neglecting all terms of order $(f_{\lambda\mu})^q$, $q \geq 2$, but to this approximation (I.28) is satisfied identically. In other words, measurements, accurate only to the first order in $f_{\lambda\mu}$, yielding the above-mentioned prediction, *are insufficient to determine whether or not the electromagnetic field of the sun satisfies* (I.28). This prediction could be tested only by a refinement of measurements that would permit the investigation of terms of the magnitude $(f_{\lambda\mu})^q$, $q \geq 2$.

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[1] *I.e.*, the Lorentz electromagnetic force vector as it is involved in the gravitational theory whenever we also consider the electromagnetic field.

This concludes the general discussion of the Principles A, B, C. A more detailed discussion will follow in the second part of this Preface, where we shall also discuss the four-component spinor algebra and analysis of the unified field.

II. WORKING SCHEME

This book contains six chapters. The first chapter deals exclusively with Principle A. The next two chapters are devoted to Principle B. In the fourth chapter we discuss the geometrical consequences of Principle C, while its physical interpretations are exhibited in the fifth chapter. The sixth chapter contains three appendices which will be described later.

The first chapter consists of fifteen sections. After the two introductory sections we discuss in the third section the tensor $*g^{\lambda\nu}$ inverse to $g_{\lambda\nu}$. These two tensors lead to basic vectors (eigenvectors of $k_\mu{}^\nu$) $\underset{1}{a}^\nu, \ldots, \underset{4}{a}^\nu$ for the first two classes (sections three and four). They are investigated in detail in sections five, six and seven. The vectors $\underset{1}{a}^\nu, \ldots, \underset{4}{a}^\nu$, together with their inverse set $\overset{1}{a}_\lambda, \ldots, \overset{4}{a}_\lambda$, constitute an intrinsic nonholonomic frame (section eight) whose elementary properties are investigated in section nine. The nonholonomic components of tensors which one obtains by means of tensorial operations from $g_{\lambda\mu}$ and $*g^{\lambda\nu}$ are introduced in section nine, while in the next section we construct by tensorial operations tensors of the type $\overset{1}{a}_\lambda \underset{1}{a}^\nu, \ldots, \overset{4}{a}_\lambda \underset{4}{a}^\nu$. These last tensors are used for tensorial construction of basic vectors by means of an arbitrary vector u^ν. In section eleven we characterize all three classes by means of elementary line geometry in P_3. This characterization is a useful tool for easy construction of the nonholonomic frame $\underset{1}{a}^\nu, \ldots, \underset{4}{a}^\nu, \overset{1}{a}_\lambda, \ldots, \overset{4}{a}_\lambda$ for the third class (sections twelve and thirteen). A very short section fourteen is devoted to the tensor $*h^{\lambda\nu} \overset{\mathrm{def}}{=} *g^{(\lambda\nu)}$, while in the last section of this chapter we introduce the tensor $f_{\lambda\mu}$, which in chapter five will be identified with the electromagnetic field.

The second chapter is divided into two parts. The *first* part, comprising eight sections, is devoted to the solution of (I.5) for general n. In the first three sections we first put (I.5) into tensorial form (I.6) and derive certain properties of the tensors $D_\omega g_{\lambda\mu}$ and $D_\omega {}^*g^{\lambda\nu}$. These results lead to (I.9) and (I.11). Before discussing the possibility of solving (I.11)a, we prove the relation $\overline{\Gamma}_\lambda{}^\nu{}_\mu = \Gamma_\mu{}^\nu{}_\lambda$ in section four. Section five is concerned with the construction of the tensor Y which occurs in the solution

$$\text{(II.1)} \qquad 2S_{\omega\mu\nu} = Y^{\alpha\beta\gamma}_{\omega\mu\nu} K_{\alpha\beta\gamma}.$$

In certain special cases the solution of (I.5) does not require direct use of the tensor Y (section six). The tensor $U^\nu{}_{\lambda\mu}$ is studied in the seventh section. For some special cases of $U^\nu{}_{\lambda\mu}$, the solution of (I.5) does not require the tensor Y. Two such cases are worth mentioning:

(1) The autoparallel lines with respect to both connections $\Gamma_\lambda{}^\nu{}_\mu$ and $\{\lambda{}^\nu{}_\mu\}$ coincide, while $S_{\lambda\mu}{}^\nu \neq 0$.

(2) Only minimal autoparallel lines of these two connections coincide, and $S_{\lambda\mu}{}^\nu \neq 0$.

In the first case the paths of particles moving freely in the unified field $g_{\lambda\mu}$ coincide with the paths of particles moving freely in the gravitational field $h_{\lambda\mu}$ of the gravitational theory. In the second case only the paths of photons are the same in both theories.

The eighth and concluding section of the first part presents a device for obtaining a rapidly converging approximation to $\Gamma_\lambda{}^\nu{}_\mu$. This device can be applied when the field $k_{\lambda\mu}$ is weak. In m steps one obtains a connection $\underset{m}{\Gamma_\lambda{}^\nu{}_\mu}$ which differs from $\Gamma_\lambda{}^\nu{}_\mu$ by terms of the order of magnitude $(k_{\lambda\mu})^q$, $q \geq 2m + 2$.

The *second* part of the second chapter is specifically concerned with the four-dimensional case. It consists of sections nine to twenty-one. After some necessary prerequisites for the first class (section nine), we introduce (section ten) the tensors $Z^{\alpha\beta\gamma}_{\omega\mu\nu}$ and $L_{\omega\mu\nu}$ by explicit definition in terms of $g_{\lambda\mu}$ and $K_{\omega\mu\nu}$. The tensor $L_{\omega\mu\nu}$ is involved in the solution (II.1) which reduces for $n = 4$ to (I.12) equivalent to

$$\text{(II.2)} \qquad 2S_{\omega\mu\nu} = L_{\omega\mu\nu} + 3L_{[\alpha\beta\nu]} k_\omega{}^\alpha k_\mu{}^\beta$$

while

$$U^\nu{}_{\lambda\mu} = -h^{\alpha\nu} k_{(\lambda}{}^\beta L_{\mu)\alpha\beta}$$

(section eleven). Comparing (II.1) and (II.2), one obtains the components Y as explicit functions of $g_{\lambda\mu}$, (section twelve). Substituting from the last two equations into (I.9) one obtains the solution $\Gamma_\lambda{}^\nu{}_\mu$ expressed explicitly in terms of $g_{\lambda\mu}$ and $\nabla_\nu g_{\lambda\mu}$. The nonholonomic components of $\Gamma_\lambda{}^\nu{}_\mu$ with respect to the intrinsic nonholonomic frame turn out to be amazingly simple (section thirteen).

The previous results for the first class are specialized in the fourteenth section for the second class.

In order to find the solution $\Gamma_\lambda{}^\nu{}_\mu$ for the third class, we first discuss some properties of the corresponding nonholonomic frame (section fifteen) which enable us to prove (sections sixteen and seventeen) that $\underset{1}{\Gamma}_{[\lambda}{}^\nu{}_{\mu]}$ [1] is the exact solution of (I.11)a for the third class. Therefore, the solution (II.1) in this case assumes a relatively simple form (I.13) (section eighteen). The next section introduces some necessary prerequisites for the twentieth section where we find necessary and sufficient conditions that (II.2), derived for the first two classes, also hold for the third class. These conditions are expressed in terms of the vectors $\underset{1}{a}^\nu, \ldots, \underset{4}{a}^\nu$, belonging to the third class, and are invariant with respect to the admissible transformations of these vectors. They are not always satisfied. Hence, we have two essentially different unified theories: one based on (II.2) for the first two classes, and the other based on (I.13) for the third class. The last section deals with some special cases of $g_{\lambda\mu}$ which occur in physical applications. This concludes the investigation of Principle B for the non-degenerate cases.

Einstein once expressed the hope that the cases in which (I.5) admits more than one solution might be physically promising. This encouraged me to deal with Principle B for the degenerate cases, $g(g-2) = 0$. This condition excludes the third class.

The results of these investigations are contained in the *third chapter*. It turns out that a tensor introduced by Nijenhuis for other purposes is a handy tool. Therefore, the first part of this chapter studies the relationship between this tensor (here denoted by $N_{\lambda\mu}{}^\nu$) and $\Gamma_\lambda{}^\nu{}_\mu$, while the second part is concerned with the applications.

The first part contains six sections. After the necessary prerequisites in the first section, we deal in the second section with the geometrical properties of $N_{\lambda\mu}{}^\nu$. In the next two sections we express

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[1] $\underset{1}{\Gamma}_{[\lambda}{}^\nu{}_{\mu]}$ is the first approximation to $\Gamma_{[\lambda}{}^\nu{}_{\mu]}$ (see the text above).

$N_{\lambda\mu}{}^{\nu}$ in terms of $S_{\lambda\mu}{}^{\nu}$ and $g_{\lambda\mu}$ and derive from this expression some identities for $N_{\lambda\mu}{}^{\nu}$ (section four). By using the condition (I.15)a we express $S_{\lambda\mu}{}^{\nu}$ in terms of $N_{\lambda\mu}{}^{\nu}$ and $g_{\lambda\mu}$ (section five), and, hence, obtain in section six the geometrical interpretation of $S_{\lambda\mu}{}^{\nu} = 0$. It turns out that this condition reduces our space-time to a Riemannian product space in which every pair of basic vectors is X_2-forming.

In the first three sections (seven, eight and nine) of the second part we study the solution of (I.11)a in the degenerate case $g = 0$. A necessary and sufficient condition, for both classes, that (I.11)a admit at least one solution is that the basic vectors $\underset{3}{a}^{\nu}$ and $\underset{4}{a}^{\nu}$ be X_2-forming. The requirement (I.15)a reduces the set of all possible solutions to a two-parameter set. This requirement does not impose any restriction on $g_{\lambda\mu}$. The concluding (tenth) section is concerned with the case $g = 2$ for the second class. A necessary and sufficient condition for the existence of at least one solution of (I.11)a is that the basic vectors $\underset{1}{a}^{\nu}$ and $\underset{2}{a}^{\nu}$ be tangent vectors of geodesic lines. In this case the condition (I.15)a imposes certain restrictions on $g_{\lambda\mu}$, one of these being that the basic vectors $\underset{1}{a}^{\nu}$ and $\underset{2}{a}^{\nu}$ be X_2-forming. In all degenerate cases the solutions are given explicitly.

In the next considerations we again assume $g(g - 2) \neq 0$.

The fourth chapter is devoted to the geometrical aspects of Principle C. It contains eleven sections. In the first three sections we deal with the curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ and its principal identities and, finally, express the tensor $R_{\mu\lambda}$ in terms of $H_{\mu\lambda}$ and $U^{\nu}{}_{\lambda\mu}$. The fourth section exhibits the uniqueness Theorem concerning $f_{\lambda\mu}$. In the next section we find two solutions of (I.15) which we use in sections six, seven and eight. In these sections the usual procedure of solving the systems (I.6) and (I.15)[1] is reversed. To this effect certain particular solutions $^{0}\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, $^{0}g_{\lambda\mu}$ are exhibited. Using $^{0}\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ one finds the general solutions $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ of (I.15) by solving a system of *algebraic equations*. Substituting $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ into (I.6) one has a system of differential equations for $g_{\lambda\mu}$. Using $^{0}g_{\lambda\mu}$ one finds the general solution of (I.6) by solving a system of *algebraic equations*. In this way the solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, $g_{\lambda\mu}$ of (I.6) and (I.15) is obtained from a particular solution $^{0}\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, $^{0}g_{\lambda\mu}$ by *algebraic procedure*. In the next three sections we study the change $'\Gamma_\lambda{}^\nu{}_\mu \rightarrow \Gamma_\lambda{}^\nu{}_\mu$ induced by a conformal transformation

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[1] By *usual procedure* we mean here first finding the solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ of (I.6) and then finding the solution $g_{\lambda\mu}$ of (I.15).

$$'g_{\lambda\mu} = e^{\Omega} g_{\lambda\mu}, \ (\Omega = \Omega(x) \neq 0). \tag{II.3}$$

An explicit expression for $'\Gamma_\lambda{}^\nu{}_\mu$ is exhibited. It leads to a corresponding change $'S_{\lambda\mu}{}^\nu \rightarrow S_{\lambda\mu}{}^\nu$ and to necessary and sufficient conditions in terms of $g_{\lambda\mu}$ for $'S_\lambda = 0$ (sections nine and ten). In the last section a connection $\tilde{\Gamma}_\lambda{}^\nu{}_\mu$ is defined in terms of $g_{\lambda\mu}$. This connection is invariant with respect to the conformal change (II.3) and is such that $\tilde{S}_\lambda = 0$.

The fifth chapter concerns the physical interpretation of Principle C. It consists of two parts. The *first* part is devoted to the third class; the *second* part deals with the first two classes. The first part contains fifteen sections. After having found the first set of Maxwell's equations (I.17)a₁, we define, in the first section, the vector v^ν by

$$v^\nu \sqrt{|\mathfrak{h}|} \overset{\text{def}}{=} \sigma \mathfrak{E}^{\nu\omega\mu\lambda} S_{\omega\mu\lambda}, \tag{II.4}$$

and express the tensors $S_{\omega\mu\nu}$ and $U^\nu{}_{\lambda\mu}$ explicitly in terms of v^ν and $f_{\lambda\mu}$ (section two). In this way we obtain (I.20) where the right hand term is an explicit function of the arguments $h_{\lambda\mu}$, $\partial_\omega h_{\lambda\mu}$, $f_{\lambda\mu}$, $\nabla_\nu f_{\lambda\mu}$ and v^ν. In the third and fourth sections we use the equation

$$R_{[\omega\mu]} = \partial_{[\omega} X_{\mu]} \tag{II.5}$$

equivalent to (I.15)b and (I.16)a. Here X_μ is an arbitrary vector. Substituting from (I.20) into (II.5), we obtain another determination for v^ν besides (II.4). These two systems lead to the second set (I.17)b of Maxwell's equations for the unknowns $f_{\lambda\mu}$. This result justifies the identification of the tensor field $f_{\lambda\mu}$ with the electromagnetic field. The above mentioned substitution from (I.20) into (II.5) results in a set of differential equations whose relationship to the Maxwell equations is studied in detail in the fifth section. The next two sections exhibit some auxiliary results needed for sections eight and nine where we obtain (I.24) from (I.15)c. These equations lead to the identification of the tensor field $h_{\lambda\mu}$ with the gravitational field. It ought to be stressed that $h_{\lambda\mu}$ has the fundamental property of a metric tensor: Every system (I.29) admits a first integral (I.30). Section ten concerns the momentum energy tensor $T_{\lambda\mu}$, the conservation law (I.25) and the equations of motion (I.26). In the eleventh section the latter equations are subjected to the criticism mentioned in the General Outline. The last three sections of this part are devoted to the consequences of (I.15)b and (I.16)b. The main results of this new approach have already been mentioned in the General Outline: the unified law of inertia (I.29); its approximations by the Newtonian law of gravitation and by the gravitational relativistic Newtonian law; its applications to celestial mechanics.

The *second* part of the fifth chapter contains sections sixteen to twenty-five. With the exception of the last section we follow here the working scheme of the first part, applied to the first two classes. In the last section of this part we investigate, from the point of view of Principle C, some special cases of the tensors $S_{\omega\mu}{}^{\nu}$ and $U^{\nu}{}_{\lambda\mu}$. Thus, for instance, if

$$S_{\omega\mu}{}^{\nu} = \tfrac{2}{3} S_{[\omega} \delta^{\nu}_{\mu]},$$

the requirement (I.15)a reduces space-time to a Riemannian space, and $k_{\lambda\mu}$ might be identified with the tensor of the electromagnetic field. Another interesting case is characterized by $U^{\nu}{}_{\lambda\mu} = 0$; here the gravitational law of inertia is the same as the unified law of inertia in spite of the fact that $S_{\omega\mu}{}^{\nu} \neq 0$. If $S_{[\omega\mu\nu]} \neq 0$ then the tensor $g_{\lambda\mu}$ is not of the first class. If

$$U^{\nu}{}_{\lambda\mu} = 2p^{\nu} h_{\lambda\mu} - \delta^{\nu}_{\lambda} p_{\mu} - \delta^{\nu}_{\mu} p_{\lambda},$$

then the tensor

$$m_{\lambda\mu} = \tfrac{1}{2}\sqrt{|\mathfrak{g}|}\, \mathfrak{e}_{\lambda\mu\alpha\beta} k^{\alpha\beta}$$

also satisfies the first set of Maxwell's equations. This concludes the investigation of Principle C.

* * *

The sixth chapter consists of three appendices whose topics are connected with theory considered in the first five chapters.

The *first appendix* is concerned with *the geometry of light signals* from the point of view of the unified theory. We imagine a light signal sent from a point O to points A, B which are situated in our observable three-dimensional space in such a way that the points O, A, B are linearly independent and $OA = OB$. We assume that

a) the gravitational field is so weak that it may be neglected,
b) the velocity of light is constant.

Denote by $t_A(t_B)$ the time interval needed for the light signal to reach the point $A(B)$. It turns out that there are fields for which $t_A \neq t_B$, as well as fields for which $t_A = t_B$. The explanation lies in the detailed investigation of (I.29) and of the tensor $U^{\nu}{}_{\lambda\mu}$. In particular, $U^{\nu}{}_{\lambda\mu}$ yields the first curvature of the trajectory of the light signal in our observable three-dimensional space. Keeping in mind the purely geometrical result

$$\text{either } t_A = t_B \text{ or } t_A \neq t_B \tag{II.9}$$

one cannot help thinking of the discrepancy between the experiments by Michelson-Morley and by Miller. In any case (II.9) shows, of course *purely geometrically*, that neither one of these experiments is decisive for or against the constancy of the velocity of light: *in the space-time endowed with a unified field, neither of the results* (II.9) *is at variance with the assumption that the velocity of light is constant.*

The second appendix deals with four-component spinor algebra and analysis of the unified field $g_{\lambda\mu}$ of the *first* class. It is divided into two parts. The first part is devoted to the spinor algebra of the local spinor space $\mathscr{S}_3(O)$; the second part concerns the spinor analysis of the spinor space $\mathscr{S}_3$.

The point of departure of the *first* part is the ideal projective space $P_3(O)$ of the linear tangent space $T_4(O)$ at a generic point O. In $P_3(O)$ we have the quadric $H(O)$ defined by $h_{\lambda\mu}$ and the nonholonomic point and plane frame $\underset{i}{a}^{\nu}$, $\overset{i}{a}_{\lambda}$. This frame is known up to transformations

$$'\underset{j}{a}^{\nu} = \overset{-1}{C_j^i}\,\underset{i}{a}^{\nu},\quad '\overset{i}{a}_{\lambda} = C_j^i\,\overset{j}{a}_{\lambda} \tag{II.6}$$

where the C's are functions of two-parameters φ, ψ. The frame $\underset{j}{a}^{\nu}$, $\overset{i}{a}_{\lambda}$ gives rise to nonholonomic coordinates of points and planes

$$v^i = v^{\lambda}\overset{i}{a}_{\lambda},\quad w_j = w_{\lambda}\underset{j}{a}^{\lambda}.$$

The relationship between $P_3(O)$ and $\mathscr{S}_3(O)$ is established by

$$v_{\mathfrak{a}}{}^{\mathfrak{b}} = E_{\lambda\mathfrak{a}}{}^{\mathfrak{b}}\,v^{\lambda} \tag{II.7}$$

where $((E_{\lambda\mathfrak{a}}{}^{\mathfrak{b}}))$ is given in terms of $\overset{i}{a}_{\lambda}$, and $v_{\mathfrak{a}}{}^{\mathfrak{b}}$ is required to be a tensor in $\mathscr{S}_3(O)$. After these prerequisites in the first two sections, we investigate, in the next four sections, the geometrical meaning of $v_{\mathrm{a}}{}^{\mathrm{b}}$, assuming first that v^i is an isotropic point (*i.e.*, a point on H). The tensor $v_{\mathrm{a}}{}^{\mathrm{b}}$ defines a degenerate projectivity in $\mathscr{S}_3(O)$. The locus of spinors having no image under this projectivity is a spinor line v^{ab} explicitly defined by means of v^i. The locus of all lines v^{ab} as v^i moves along H is a linear line congruence $\mathscr{H}$ in $\mathscr{S}_3(O)$. These results lead in the fifth section to the coefficients $\mathscr{C}_{\mathrm{a}}{}^{\mathrm{b}}$ of the spinor transformation in $\mathscr{S}_3(O)$

$$(\mathrm{II}.8) \qquad {}'\mathscr{x}^{\mathrm{b}} = \mathscr{C}_{\mathrm{a}}{}^{\mathrm{b}}\mathscr{x}^{\mathrm{a}}$$

induced by (II.6) in $P_3(O)$. The $\mathscr{C}$'s are functions of the same arguments as the C's. There are always two transformations (II.8) corresponding to each transformation (II.6). If the point v^i is not isotropic, then $v_{\mathrm{a}}{}^{\mathrm{b}}$ defines a biaxial involution whose axes ${}^{\epsilon}v^{\mathrm{ab}}$ ($\epsilon = \pm 1$) are explicitly given by means of v^i (section six). The locus of all lines as v^i moves along $P_3(O)$ is a linear complex k_{ab} containing the congruence $\mathscr{H}$ (section seven). In the eighth section we prove that k_{ab} are components of $k_{\mu\lambda}$ with respect to a particular nonholonomic frame. The biaxial involution $v_{\mathrm{a}}{}^{\mathrm{b}}$ in $\mathscr{S}_3(O)$ is represented in $P_3(O)$ by a v^{α}-reflection (section nine). The tenth section develops some necessary prerequisites for the investigation of lineal elements (*i.e.*, an isotropic point v^i and a tangent to H at v^i). Such a lineal element in $P_3(O)$ is mapped into $\mathscr{S}_3(O)$ on two spinors whose coordinates are explicitly given in terms of the lineal element (section eleven). Conversely, a given spinor in $\mathscr{S}_3(O)$ is a map of a lineal element which is explicitly given by means of the coordinates of this spinor. This mapping also involves the aforementioned complex k_{ab} (section twelve). In the thirteenth and fourteenth sections we investigate Dirac's equations and Schrödinger's wave equation by means of a tensor ${}_{*}v_{\mathrm{a}}{}^{\mathrm{b}}$ whose matrix is similar to $2((v_{\mathrm{a}}{}^{\mathrm{b}}))$. The tensor $v_{\mathrm{a}}{}^{\mathrm{b}}$ is defined by (II.7), and the (symbolical) point v^i is a combination of the components of the linear differential operator in the quantum theory. It turns out that ${}_{*}v_{\mathrm{a}}{}^{\mathrm{b}}$ gives rise to a (symbolical) biaxial involution, and Dirac's equations define its eigenspinors, *i.e.*, the spinors incident with the axes ${}_{*}{}^{\epsilon}v^{\mathrm{ab}}$ of ${}_{*}{}^{\epsilon}v_{\mathrm{a}}{}^{\mathrm{b}}$. The corresponding Schrödinger equation can be derived from the corresponding ${}_{*}v^{\alpha}$-reflection. This equation defines a scalar function of double elements of the reflection.

The second part of Appendix II starts with the projection of $\mathscr{S}_3(U)$ onto $\mathscr{S}_3(O)$ (section fifteen). This projection gives rise to a trace-free local connection $\Pi_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda}(O)$ which is uniquely defined by $g_{\lambda\mu}$. This connection does not discard the relationship (II.7) (sections sixteen and seventeen). In the eighteenth section we use the fact that the axes of $\mathscr{H}(O)$ do not depend on O, and thus, enable us to superimpose the spinor space $\mathscr{S}_3(U)$ on $\mathscr{S}_3(O)$, $U \neq O$. The totality of all superimposed spaces on $\mathscr{S}_3(O)$ is by definition the spinor space $\mathscr{S}_3$ adjoint to the space-time. It is endowed with a trace-free connection $\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda}$ uniquely defined by $g_{\lambda\mu}$. This connection does not discard the relationship (II.7). The curvature tensor $\mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}}$ of $\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda}$ is expressible in terms of $H_{\omega\mu\lambda}{}^{\nu}$. Its vanishing is a necessary but not a sufficient condition for $\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda} = 0$. The electromagnetic field of $g_{\lambda\mu}$ does not generate gravitation if and only if $\mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}} = 0$ (section nineteen). In the last section we establish a set of equations in $\mathscr{S}_3$ which reduce for $\mathscr{S}_3(O)$ to Dirac's equations.

The third appendix is provided for the convenience of those not familiar with the algebra of line geometry. It presents some elementary results in this geometry needed for the understanding of certain sections of this book.

III. NOTATION

Throughout this book extensive use is made of Tensor Calculus in an n-dimensional space. We always use Greek covariant and contravariant indices to express tensors by means of their components with respect to an arbitrary coordinate system. Their range is the set of the first n Roman numerals. Thus, if $n = 4$, the range of Greek indices is I, II, III, IV. Greek indices always follow the summation convention.

We use Roman subscripts and superscripts to express tensors by means of their components with respect to a nonholonomic frame of reference. Unless the contrary is explicitly stated, these indices run from 1 to n. They follow the summation convention unless the contrary is stated, *with the exception of indices x, y, z, t. The summation for these indices is always denoted by the sign Σ.*

Let $T_{\lambda\ldots\mu}$ be a set of terms with p indices $\lambda\ldots\mu$. Denote by $E_{\lambda\ldots\mu}(O_{\lambda\ldots\mu})$ the sum of all T's with even (odd) permutation of the indices. Then the operation of *mixing* over p indices or of *alternating* over p indices is defined as follows:

$$p!T_{(\lambda\ldots\mu)} \overset{\text{def}}{=} E_{\lambda\ldots\mu} + O_{\lambda\ldots\mu}$$

$$p!T_{[\lambda\ldots\mu]} \overset{\text{def}}{=} E_{\lambda\ldots\mu} - O_{\lambda\ldots\mu}.$$

Thus,

$$6T_{(\lambda\omega\mu)} = (T_{\lambda\omega\mu} + T_{\omega\mu\lambda} + T_{\mu\lambda\omega}) + (T_{\omega\lambda\mu} + T_{\lambda\mu\omega} + T_{\mu\omega\lambda})$$

$$6T_{[\lambda\omega\mu]} = (T_{\lambda\omega\mu} + T_{\omega\mu\lambda} + T_{\mu\lambda\omega}) - (T_{\omega\lambda\mu} + T_{\lambda\mu\omega} + T_{\mu\omega\lambda}).$$

To exclude certain indices from mixing or alternating we enclose them by | |. Examples:

$$2T_{(\lambda|\omega|\mu)} = T_{\lambda\omega\mu} + T_{\mu\omega\lambda}$$

$$2T_{[\lambda|\omega|\mu]} = T_{\lambda\omega\mu} - T_{\mu\omega\lambda}.$$

By ∂_λ we always mean $\frac{\partial}{\partial x^\lambda}$, x^λ being coordinates of the space. Unless the contrary is explicitly stated the radical sign $\sqrt{\ }$ or $(\ )^{\frac{1}{2}}$ always represents a positive square root. To facilitate cross references we use the following rule: references not preceded by a chapter number refer to the chapter in which they occur.

Chapter I

ALGEBRA OF THE UNIFIED TENSOR FIELD $g_{\lambda\mu}$

Introduction. Denote by x^ν a real coordinate system of an n-dimensional space X_n. Starting with x^ν we may consider transformations of real coordinate systems. We shall confine ourselves to transformations for which

$$\Delta \overset{\text{def}}{=} Det\left(\left(\frac{\partial x^{\nu'}}{\partial x^\lambda}\right)\right) \neq 0. \tag{1}$$

A tensor of contravariant valence p and covariant valence q is characterized by the transformation rule of its components $T^{\nu_1\dots\nu_p}_{\lambda_1\dots\lambda_q}$

$$T^{\nu_1'\dots\nu_p'}_{\lambda_1'\dots\lambda_q'} = \frac{\partial x^{\nu_1'}}{\partial x^{\alpha_1}} \cdots \frac{\partial x^{\nu_p'}}{\partial x^{\alpha_p}} \, \frac{\partial x^{\beta_1}}{\partial x^{\lambda_1'}} \cdots \frac{\partial x^{\beta_q}}{\partial x^{\lambda_q'}} \, T^{\alpha_1\dots\alpha_p}_{\beta_1\dots\beta_q}. \tag{2}$$

If $p = 1$, $q = 0$ ($p = 0$, $q = 1$) the tensor is termed a *contravariant* (a *covariant*) *vector.* If $p = q = 0$, the tensor is called a *scalar.* Tensors will be denoted by symbols in Roman type.

A *density* $\mathfrak{a}$ *of weight* w is characterized by its rule of transformation

$$'\mathfrak{a} = \Delta^{-w}\mathfrak{a}. \tag{3}$$

The product of a density of weight w and a tensor will be termed *a tensor density of weight w.* Densities and tensor-densities will be denoted by symbols in Gothic type.

Let $\mathfrak{e}$ stand for an arbitrary density of weight $w = +1$ or $w = -1$. Then the number

$$\sigma \overset{\text{def}}{=} \mathfrak{e}/|\mathfrak{e}| \tag{4}$$

transforms according to

$$'\sigma = \epsilon\sigma \tag{5}$$

$$(\epsilon \overset{\text{def}}{=} \operatorname{sgn} \Delta).$$

The product of a tensor and σ will be termed a *pseudotensor.* No special notation will be used for pseudotensors. If T^ν is a contravariant pseudovector the transformation rule for its components is

$$T^{\nu'} = \epsilon \frac{\partial x^{\nu'}}{\partial x^{\lambda}} T^{\lambda}. \tag{6}$$

The product of a density (of weight w) and σ will be termed a *pseudodensity* (of weight w). Pseudodensities will be denoted by symbols in gothic type. If $\mathfrak{a}$ is a pseudodensity of weight w its rule of transformation is

$$'\mathfrak{a} = \epsilon \Delta^{-w} \mathfrak{a}. \tag{7}$$

Hence, if $\mathfrak{m}$ is a density of weight -1, then $|\mathfrak{m}|$ is a pseudodensity, because

$$|'\mathfrak{m}| = |\Delta|\,|\mathfrak{m}| = \epsilon \Delta |\mathfrak{m}|.$$

Of course, $\sigma|\mathfrak{m}|$ is then again a density.

1. The tensor $g_{\lambda\mu}$. Throughout this chapter a four dimensional space X_4 referred to a coordinate system x^{ν} will be considered. We shall admit only coordinate transformations $x^{\nu'} \leftrightarrow x^{\nu}$ for which

$$\Delta \stackrel{\text{def}}{=} \begin{vmatrix} \frac{\partial x^{I'}}{\partial x^{I}} & \cdots & \frac{\partial x^{I'}}{\partial x^{IV}} \\ \vdots & & \vdots \\ \frac{\partial x^{IV'}}{\partial x^{I}} & \cdots & \frac{\partial x^{IV'}}{\partial x^{IV}} \end{vmatrix} \neq 0. \tag{1.1}$$

All coordinate systems are assumed to be real.

The space X_4 is endowed with a real [1] quadratic tensor $g_{\lambda\mu}$. It may be decomposed into its symmetric part $h_{\lambda\mu}$ and its skew symmetric part $k_{\lambda\mu}$

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{1.2a}$$

$$h_{\lambda\mu} \stackrel{\text{def}}{=} g_{(\lambda\mu)} = \tfrac{1}{2}(g_{\lambda\mu} + g_{\mu\lambda}) \tag{1.2b}$$

$$k_{\lambda\mu} \stackrel{\text{def}}{=} g_{[\lambda\mu]} = \tfrac{1}{2}(g_{\lambda\mu} - g_{\mu\lambda}). \tag{1.2c}$$

AGREEMENT (1.1). *In all that follows in this book* [2] *we assume that*

a) *the determinant* $\mathfrak{g}$ *of* $g_{\mu\lambda}$ *is negative*

$$\mathfrak{g} < 0, \tag{1.3a}$$

---

[1] In his earlier papers, Einstein considered a complex tensor $g_{\lambda\mu}$. (*Cf.* Einstein, A., (1) and Einstein, A.-Strauss, E., (1).

[2] With the exception of Chapter III.

b) *the tensor* $h_{\lambda\mu}$ *has the signature* $(+++-)$, *so that its determinant* $\mathfrak{h}$ *is negative*

$$\mathfrak{h} < 0, \tag*{(1.3)b}$$

c) *the tensor* $h_{\lambda\mu}$ *is the metric tensor,*
d) *the tensor* $k_{\lambda\mu}$ *is always present*

$$k_{\lambda\mu} \neq 0. \tag*{(1.3)c}$$

REMARK I: According to (1.3)b there is a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ defined by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^{\nu}_{\mu}.\,^{1} \tag*{(1.4)}$$

The tensor $h^{\lambda\nu}(h_{\lambda\mu})$ will serve for raising (lowering) indices in the usual manner [2].

REMARK II: According to (1.2)c the determinant $\mathfrak{k}$ of $k_{\lambda\mu}$ is never negative

$$\mathfrak{k} \geq 0. \tag*{(1.3)d}$$

REMARK III: Each of the determinants $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ is a density of weight 2. However, $\sqrt{|\mathfrak{g}|}$, $\sqrt{|\mathfrak{h}|}$, and $\sqrt{\mathfrak{k}}$ are pseudodensities of weight 1. Thus, for instance, [3]

$$\sqrt{|'\mathfrak{h}|} = |\Delta|^{-1}\sqrt{|\mathfrak{h}|} = \epsilon\Delta^{-1}\sqrt{|\mathfrak{h}|}$$
$$(\epsilon \overset{\text{def}}{=} \operatorname{sgn} \Delta).$$

In subsequent considerations we shall need the following scalars

$$g \overset{\text{def}}{=} \mathfrak{g}/\mathfrak{h} > 0 \tag*{(1.5)a}$$

$$k \overset{\text{def}}{=} \mathfrak{k}/\mathfrak{h} \leq 0 \tag*{(1.5)b}$$

$$4K \overset{\text{def}}{=} k_{\alpha\beta}k_{\gamma\delta}h^{\alpha\gamma}h^{\beta\delta} = k_{\alpha\beta}k^{\alpha\beta}. \tag*{(1.5)c}$$

DEFINITION (1.1). *The tensor* $k_{\lambda\mu}$ *will be said to be*

---

[1] δ^{ν}_{μ} is the Kronecker delta. Later in this study we shall also use the abbreviation

$$\delta^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_p} \overset{\text{def}}{=} \delta^{\nu_1}_{\lambda_1}\cdots\delta^{\nu_p}_{\lambda_p}.$$

[2] We write δ^{ν}_{μ} instead of $h_{\mu}{}^{\nu}$ for historical reasons.

[3] The square root is always to be taken as positive.

1) *of the first class whenever*

(1.6)a $$k \neq 0,$$

2) *of the second class whenever*

(1.6)b $$K \neq 0,\ k = 0,$$

3) *of the third class whenever*

(1.6)c $$K = k = 0.$$ [1]

REMARK. Unless explicitly stated otherwise every result of our subsequent considerations holds for all three classes.

2. The indicators $\mathfrak{G}^{\omega\mu\lambda\nu}$, $\mathfrak{e}_{\omega\mu\lambda\nu}$.

DEFINITION (2.1). *The* $\begin{cases}\textit{contravariant}\\ \textit{covariant}\end{cases}$ *indicator* $\begin{cases}\mathfrak{G}^{\omega\mu\lambda\nu}\\ \mathfrak{e}_{\omega\mu\lambda\nu}\end{cases}$ *is a tensor density of weight* $\begin{cases}1\\ -1\end{cases}$, *skew symmetric in all indices, whose components* $\begin{cases}\mathfrak{G}^{\omega\mu\lambda\nu}\\ \mathfrak{e}_{\omega\mu\lambda\nu}\end{cases}$ *have (in all coordinate systems) the following numerical values*

a) $+\ 1$ *whenever* $\omega\mu\lambda\nu$ *is an even permutation of I, II, III, IV*

b) $-\ 1$ *whenever* $\omega\mu\lambda\nu$ *is an odd permutation of I, II, III, IV*

c) 0 *in all remaining cases.*

REMARK. According to this definition the transformation law for indicators is

(2.1)
$$\begin{aligned}\mathfrak{G}^{\omega'\mu'\lambda'\nu'} &= \varDelta^{-1}\mathfrak{G}^{\alpha\beta\gamma\delta}\,\frac{\partial x^{\omega'}}{\partial x^{\alpha}}\,\frac{\partial x^{\mu'}}{\partial x^{\beta}}\,\frac{\partial x^{\lambda'}}{\partial x^{\gamma}}\,\frac{\partial x^{\nu'}}{\partial x^{\delta}}\\ \mathfrak{e}_{\omega'\mu'\lambda'\nu'} &= \varDelta\,\mathfrak{e}_{\alpha\beta\gamma\delta}\,\frac{\partial x^{\alpha}}{\partial x^{\omega'}}\,\frac{\partial x^{\beta}}{\partial x^{\mu'}}\,\frac{\partial x^{\gamma}}{\partial x^{\lambda'}}\,\frac{\partial x^{\delta}}{\partial x^{\nu'}}.\end{aligned}$$

The indicators are related in a way described in the following

THEOREM (2.1). *We have:*

(2.2)a $$\mathfrak{G}^{\alpha\beta\gamma\delta}\mathfrak{e}_{\alpha\beta\gamma\delta} = 4!$$

(2.2)b $$\mathfrak{G}^{\alpha\beta\gamma\nu}\mathfrak{e}_{\alpha\beta\gamma\lambda} = 3!\delta^{\nu}_{\lambda}$$

(2.2)c $$\mathfrak{G}^{\alpha\beta\lambda\nu}\mathfrak{e}_{\alpha\beta\omega\mu} = 4\delta^{\lambda\,\nu}_{[\omega\mu]}$$

(2.2)d $$\mathfrak{G}^{\alpha\mu\lambda\nu}\mathfrak{e}_{\alpha\xi\eta\zeta} = 3!\delta^{\mu\lambda\nu}_{[\xi\eta\zeta]}$$

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[1] *Cf.* Hlavatý, V., (6), (15), (19); Hlavatý, V.–Sáenz, A., (12); Sáenz, A., (1).

and

$$\mathfrak{E}_{\omega\mu\lambda\nu} = \mathfrak{h}\mathfrak{e}_{\omega\mu\lambda\nu} \tag{2.3)a}$$

$$\mathfrak{e}^{\omega\mu\lambda\nu} = \mathfrak{E}^{\omega\mu\lambda\nu}/\mathfrak{h}. \tag{2.3)b}$$

PROOF. The equation (2.2)a is obvious. In order to prove (2.3), start with the obvious equations

$$4!\mathfrak{h} = \mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{E}^{\alpha\beta\gamma\delta}h_{\omega\alpha}h_{\mu\beta}h_{\lambda\gamma}h_{\nu\delta} \tag{2.4}$$

and

$$\mathfrak{E}_{\omega\mu\lambda\nu} = \mathfrak{E}^{\alpha\beta\gamma\delta}h_{\omega\alpha}h_{\mu\beta}h_{\lambda\gamma}h_{\nu\delta}. \tag{2.5}$$

According to (2.5) we must have

$$\mathfrak{E}_{\omega\mu\lambda\nu} = x\mathfrak{e}_{\omega\mu\lambda\nu} \tag{2.6}$$

where x is to be found. Substituting from (2.5) and (2.6) into (2.4), we obtain $x = \mathfrak{h}$. Hence, (2.3)a holds. The equation (2.3)b may be proved in a similar manner. On the other hand, we obtain from (1.4)

$$4!\,\frac{\partial\mathfrak{h}}{\partial h_{\lambda\nu}} = 4!\mathfrak{h}h^{\lambda\nu} \tag{2.7)a}$$

and from (2.4) and (2.3)

$$\begin{aligned} 4!\,\frac{\partial\mathfrak{h}}{\partial h_{\lambda\nu}} &= 4\mathfrak{E}^{\lambda\alpha\beta\gamma}\mathfrak{E}^{\nu\xi\eta\zeta}h_{\alpha\xi}h_{\beta\eta}h_{\gamma\zeta}\ [1] \\ &= 4\mathfrak{h}\mathfrak{E}^{\lambda\alpha\beta\gamma}\mathfrak{e}_{\omega\alpha\beta\gamma}h^{\omega\nu}, \end{aligned} \tag{2.7)b}$$

so that

$$3!h^{\lambda\nu} = \mathfrak{E}^{\lambda\alpha\beta\gamma}\mathfrak{e}_{\omega\alpha\beta\gamma}h^{\omega\nu}.$$

This equation leads at once to (2.2)b. The equation (2.2)c follows from the obvious relation

$$\mathfrak{E}^{\alpha\beta\lambda\nu}\mathfrak{e}_{\alpha\beta\omega\mu} = y\delta^{\lambda\nu}_{[\omega\mu]},$$

which leads by virtue of (2.2)b to

$$\mathfrak{E}^{\alpha\beta\gamma\nu}\mathfrak{e}_{\alpha\beta\gamma\mu} = 3!\delta^{\nu}_{\mu} = \tfrac{3}{2}\delta^{\nu}_{\mu}y,$$

so that $y = 4$. The equation (2.2)d may be proved in a similar manner.

---

[1] *Cf.* (2.9)d, page 6, setting $l_{\mu\lambda} = h_{\mu\lambda}$.

The density $4!\mathfrak{h}$ may be written in the form (2.4). A similar obvious equation holds for the determinant $\mathfrak{k}$

$$4!\mathfrak{k} = \mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{E}^{\alpha\beta\gamma\delta}k_{\omega\alpha}k_{\mu\beta}k_{\lambda\gamma}k_{\nu\delta}. \tag{2.8}$$

There are also some less obvious relations between the indicators and $h_{\lambda\mu}$, $k_{\lambda\mu}$:

THEOREM (2.2) *We have*

$$\mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{E}^{\alpha\beta\gamma\delta}h_{\omega\alpha}h_{\mu\beta}h_{\lambda\gamma}k_{\nu\delta} = 0 \tag{2.9)a}$$

$$\mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{E}^{\alpha\beta\gamma\delta}h_{\omega\alpha}h_{\mu\beta}k_{\lambda\gamma}k_{\nu\delta} = 8\mathfrak{h}K \tag{2.9)b}$$

$$\mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{E}^{\alpha\beta\gamma\delta}h_{\omega\alpha}k_{\mu\beta}k_{\lambda\gamma}k_{\nu\delta} = 0. \tag{2.9)c}$$

PROOF. Denote by $l_{\mu\lambda}$ either one of the tensors $h_{\lambda\mu}$ and $k_{\lambda\mu}$. Then

$$\begin{aligned}\mathfrak{E}^{\nu\omega\mu\lambda}\mathfrak{E}^{\alpha\beta\gamma\delta}l_{\omega\beta}l_{\mu\gamma}l_{\lambda\delta} &= \mathfrak{E}^{\alpha\omega\mu\lambda}\mathfrak{E}^{\nu\beta\gamma\delta}l_{\beta\omega}l_{\gamma\mu}l_{\delta\lambda} = \\ &= \psi\mathfrak{E}^{\alpha\omega\mu\lambda}\mathfrak{E}^{\nu\beta\gamma\delta}l_{\omega\beta}l_{\mu\gamma}l_{\lambda\delta}\end{aligned} \tag{2.9)d}$$

where

$$\psi = \begin{cases} +1 \\ -1 \end{cases} \text{for } l_{\mu\lambda} = \begin{cases} h_{\mu\lambda} \\ k_{\mu\lambda}. \end{cases}$$

Hence, if $l_{\mu\lambda} = h_{\mu\lambda}$ ($l_{\mu\lambda} = k_{\mu\lambda}$), then the scalar product of the first term of (2.9)d and a skew symmetric (symmetric) tensor is equal to zero. This proves (2.9)a and (2.9)c. The left hand term of (2.9)b may be rewritten by virtue of (2.2)c and (2.3):

$$\mathfrak{h}\mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{e}_{\omega\mu\alpha\beta}h^{\alpha\gamma}h^{\beta\delta}k_{\lambda\gamma}k_{\nu\delta} = 4\mathfrak{h}\delta^{\lambda\nu}_{[\alpha\beta]}k_{\lambda}{}^{\alpha}k_{\nu}{}^{\beta} = -2\mathfrak{h}k_{\beta}{}^{\alpha}k_{\alpha}{}^{\beta} = 8\mathfrak{h}K.$$

THEOREM (2.3). *We have*

$$64\mathfrak{k} = (\mathfrak{E}^{\omega\mu\lambda\nu}k_{\omega\mu}k_{\lambda\nu})^2. \tag{2.10}$$

PROOF. The equation (2.8) is equivalent to

$$\mathfrak{k} = (k_{I\,II}k_{III\,IV} + k_{II\,III}k_{I\,IV} + k_{III\,I}k_{II\,IV})^2,$$

while

$$\mathfrak{E}^{\omega\mu\lambda\nu}k_{\omega\mu}k_{\lambda\nu} = 8(k_{I\,II}k_{III\,IV} + k_{II\,III}k_{I\,IV} + k_{III\,I}k_{II\,IV}).$$

Comparing these two equations we obtain (2.10).

There are some pseudotensors intimately connected with the indicators which we shall need later on:

THEOREM (2.4). *Put*

$$(2.11)\text{a} \qquad E^{\omega\mu\lambda\nu} \overset{\text{def}}{=} \mathfrak{E}^{\omega\mu\lambda\nu}/\sqrt{|\mathfrak{h}|}$$

$$(2.11)\text{b} \qquad e_{\omega\mu\lambda\nu} \overset{\text{def}}{=} \mathfrak{e}_{\omega\mu\lambda\nu}\sqrt{|\mathfrak{h}|}.$$

Then $E^{\omega\mu\lambda\nu}$, $e_{\omega\mu\lambda\nu}$ *are pseudotensors which satisfy the following conditions*:

$$(2.12)\text{a} \qquad E^{\alpha\beta\gamma\delta}e_{\alpha\beta\gamma\delta} = 4!$$

$$(2.12)\text{b} \qquad E^{\alpha\beta\gamma\nu}e_{\alpha\beta\gamma\lambda} = 3!\delta^{\nu}_{\lambda}$$

$$(2.12)\text{c} \qquad E^{\alpha\beta\omega\mu}e_{\alpha\beta\lambda\nu} = 4\delta^{\omega\mu}_{[\lambda\nu]}$$

$$(2.12)\text{d} \qquad E^{\alpha\mu\lambda\nu}e_{\alpha\beta\gamma\delta} = 3!\delta^{\mu\lambda\nu}_{[\beta\gamma\delta]}$$

and

$$(2.13)\text{a} \qquad E_{\omega\mu\lambda\nu} = -e_{\omega\mu\lambda\nu}$$

$$(2.13)\text{b} \qquad e^{\omega\mu\lambda\nu} = -E^{\omega\mu\lambda\nu}.$$ [1]

PROOF. We know (*Cf.* Remark III preceding Definition (1.1)) that $\sqrt{|\mathfrak{h}|}$ is a pseudodensity of weight 1. Therefore, $E^{\omega\mu\lambda\nu}$ ($e_{\omega\mu\lambda\nu}$), as defined by (2.11)a, ((2.11)b) is a pseudotensor. The proof of (2.12) and (2.13) follows easily from (2.11) and from Theorem (2.1).

3. The tensor $^{*}g^{\lambda\nu}$**.** Denote by $\mathfrak{a}(\lambda)$ the determinant of the tensor

$$(3.1) \qquad a_{\omega\mu}(\lambda) \overset{\text{def}}{=} \lambda h_{\omega\mu} + k_{\omega\mu}$$

where λ is an arbitrary parameter.

THEOREM (3.1). *We have*

$$(3.2) \qquad \mathfrak{a}(\lambda) = \lambda^4\mathfrak{h} + 2\lambda^2\mathfrak{h}K + \mathfrak{k},$$

and in particular

$$(3.3)\text{a} \qquad \mathfrak{a}(1) = \mathfrak{g} = \mathfrak{h} + 2\mathfrak{h}K + \mathfrak{k},$$

so that

$$(3.3)\text{b} \qquad g = 1 + 2K + k.$$

PROOF. The determinant $\mathfrak{a}$ is given by

$$(3.4) \qquad 4!\mathfrak{a}(\lambda) = \mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{E}^{\alpha\beta\gamma\delta}a_{\omega\alpha}a_{\mu\beta}a_{\lambda\gamma}a_{\nu\delta}.$$

---

[1] The minus signs in (2.13) come from (1.3)b.

Substituting from (3.1) into (3.4) and remembering (2.4), (2.8) and (2.9), we obtain (3.2). The remaining part of the proof is obvious.

REMARK. Using (2.9)b and (2.10) we may express $\mathfrak{g}$ given by (3.3)a in terms of $h_{\lambda\nu}$ and $k_{\lambda\nu}$:

$$\mathfrak{g} = \mathfrak{h} + \tfrac{1}{4}\mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{E}^{\alpha\beta\gamma\delta}h_{\omega\alpha}h_{\mu\beta}k_{\lambda\gamma}k_{\nu\delta} + \tfrac{1}{64}(\mathfrak{E}^{\omega\mu\lambda\nu}k_{\omega\mu}k_{\lambda\nu})^2. \tag{3.3)c}$$

This formula will be very useful later on.

According to (1.3)a there is a unique tensor

$$*g^{\lambda\nu} \overset{\text{def}}{=} \frac{\partial \ln \mathfrak{g}}{\partial g_{\lambda\nu}} \tag{3.5}$$

satisfying the condition

$$g_{\lambda\mu}{*g^{\lambda\nu}} = g_{\mu\lambda}{*g^{\nu\lambda}} = \delta^{\nu}_{\mu}. \tag{3.6}$$

Following the pattern of (1.2) we decompose $*g^{\lambda\nu}$ into its symmetric part $*h^{\lambda\nu}$ and skew symmetric part $*k^{\lambda\nu}$:

$$*g^{\lambda\nu} = *h^{\lambda\nu} + *k^{\lambda\nu} \tag{3.7)a}$$

$$*h^{\lambda\nu} \overset{\text{def}}{=} *g^{(\lambda\nu)} \tag{3.7)b}$$

$$*k^{\lambda\nu} \overset{\text{def}}{=} *g^{[\lambda\nu]}. \tag{3.7)c}$$

In order to express $*h^{\lambda\nu}$ and $*k^{\lambda\nu}$ in terms of $g_{\lambda\nu}$ we need the tensors ${}^{(p)}k_{\lambda}{}^{\nu}$ defined as follows

$$\begin{matrix} {}^{(1)}k_{\lambda}{}^{\nu} \overset{\text{def}}{=} k_{\lambda}{}^{\nu} \\ {}^{(p)}k_{\lambda}{}^{\nu} \overset{\text{def}}{=} {}^{(p-1)}k_{\lambda}{}^{\alpha}k_{\alpha}{}^{\nu} \end{matrix}, \quad p = 2, 3, \ldots. \tag{3.8}$$

A simple inspection shows that ${}^{(p)}k^{\lambda\nu}$ is symmetric (skew symmetric) for any even p (odd p).

THEOREM (3.2). *The tensors* $*h^{\lambda\nu}$, $*k^{\lambda\nu}$ *satisfy the following equations*:

(3.9)a[1]
$$*h^{\lambda\nu} = \frac{1}{g}[h^{\lambda\nu}(1 + 2K) + {}^{(2)}k^{\lambda\nu}]$$

$$*k^{\lambda\nu} = \frac{1}{\mathfrak{g}}\left[\mathfrak{h}k^{\lambda\nu} + \frac{\kappa}{2}\sqrt{\mathfrak{k}}\,\mathfrak{E}^{\omega\mu\lambda\nu}k_{\omega\mu}\right] \tag{3.9)b}$$

$$(\kappa \overset{\text{def}}{=} \operatorname{sgn} \mathfrak{E}^{\omega\mu\lambda\nu}k_{\omega\mu}k_{\lambda\nu}).$$

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[1] *Cf.* Schrödinger, E., (1); Hlavatý, V., (16); Lichnerowicz, A., (3); Tonnelat, M. A., (13).

Proof. Starting with the obvious equation

$$4!\mathfrak{g} = \mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{E}^{\alpha\beta\gamma\delta}g_{\omega\alpha}\,g_{\mu\beta}\,g_{\lambda\gamma}\,g_{\nu\delta} \tag{3.10}$$

we obtain according to (3.5)

$$4!\frac{\partial\mathfrak{g}}{\partial g_{\lambda\nu}} = 4\mathfrak{E}^{\lambda\omega\mu\rho}\mathfrak{E}^{\nu\alpha\beta\gamma}g_{\omega\alpha}\,g_{\mu\beta}\,g_{\rho\gamma} = 4!\ \mathfrak{g}\ {}^*g^{\lambda\nu}. \tag{3.11}$$

On the other hand, if we substitute from (1.2)a into (3.10), we obtain

$$4!\,\frac{\partial\mathfrak{g}}{\partial h_{\lambda\nu}} = 4\mathfrak{E}^{(\lambda|\omega\mu\rho|}\mathfrak{E}^{\nu)\alpha\beta\gamma}g_{\omega\alpha}g_{\mu\beta}g_{\rho\gamma}. \tag{3.12}$$

Comparing (3.11) and (3.12) we have

$$\frac{\partial\ln\mathfrak{g}}{\partial\,h_{\lambda\nu}} = {}^*g^{(\lambda\nu)} = {}^*h^{\lambda\nu}. \tag{3.13)a}$$

The equation

$$\frac{\partial\ln\mathfrak{g}}{\partial\,k_{\lambda\nu}} = {}^*g^{[\lambda\nu]} = {}^*k^{\lambda\nu} \tag{3.13)b}$$

may be proved in a similar manner. The equations (3.13), (3.3)c, (2.4), (2.9)c and (2.10) yield (3.9).

REMARK I. From the definition of the sign κ we see that its transformation rule is

$$\kappa' = \epsilon\kappa\quad(\epsilon = \operatorname{sgn}\varDelta).$$

Consequently, $\kappa\sqrt{\mathfrak{k}}$ is a density of weight 1 (while $\sqrt{\mathfrak{k}}$ is a pseudo-density of weight 1). Therefore, it is obvious that ${}^*k^{\lambda\nu}$, as defined by (3.9)b, is a tensor.

REMARK II. The tensors ${}^*h^{\lambda\nu}$ and ${}^*k^{\lambda\nu}$ will play an important role in subsequent chapters. This is true, in particular, of the tensor ${}^*k^{\lambda\nu}$.

THEOREM (3.3). *The tensor* ${}^*k^{\lambda\nu}$ *never vanishes*

$${}^*k^{\lambda\nu} \neq 0. \tag{3.14}$$

PROOF. Assume ${}^*k^{\lambda\nu} = 0$. By (3.9)b, this requirement is equivalent to

$$\mathfrak{h}k^{\lambda\nu} = -\frac{\kappa}{2}\sqrt{\mathfrak{k}}\mathfrak{E}^{\alpha\beta\lambda\nu}k_{\alpha\beta}. \tag{3.15}$$

Multiplying (3.15) by $k_{\lambda\nu}$ we obtain by (1.5)c and (2.10)

$$\mathfrak{h}K = -\,\mathfrak{k}. \tag{3.16}$$

On the other hand, multiplying (3.15) by $\mathfrak{e}_{\omega\mu\lambda\nu}$ we have

$$\mathfrak{h}\, k^{\lambda\nu}\mathfrak{e}_{\omega\mu\lambda\nu} = -\,2\kappa\sqrt{\mathfrak{k}}\,k_{\omega\mu},$$

and this equation is equivalent to

(3.17) $$\kappa\sqrt{\mathfrak{k}}\,k^{\lambda\nu} = -\,\tfrac{1}{2}\mathfrak{E}^{\lambda\nu\alpha\beta}k_{\alpha\beta}.$$

Hence, we have according to (3.15) and (3.17)

(3.18) $$\mathfrak{h} = \mathfrak{k},$$

so that by virtue of (3.3)a, (3.16) and (3.18)

$$\mathfrak{g} = \mathfrak{h} + 2\mathfrak{h}K + \mathfrak{k} = 0.$$

This equation, obtained as a consequence of ${}^*k^{\lambda\nu} = 0$, contradicts (1.3)a. Therefore, (3.14) holds.

4. The tensors $T_\lambda{}^\nu$ and $\overset{-1}{T}_\lambda{}^\nu$. The tensors $g_{\lambda\mu}$ and ${}^*g^{\lambda\nu}$ are not symmetric, and therefore, we have to consider, besides $g_{\lambda\mu}{}^*g^{\lambda\nu}$ involved in (3.6), two other products, namely $g_{\lambda\mu}{}^*g^{\nu\lambda}$ and $g_{\mu\lambda}{}^*g^{\lambda\nu}$.

THEOREM (4.1). *The tensors*

(4.1)a $$T_\mu{}^\nu \overset{\text{def}}{=} g_{\lambda\mu}{}^*g^{\nu\lambda}$$

(4.1)b $$\overset{-1}{T}_\mu{}^\nu \overset{\text{def}}{=} g_{\mu\lambda}{}^*g^{\lambda\nu}$$

are both of the same rank 4 *and satisfy the following equation*:

(4.2) $$T_\mu{}^\beta\,\overset{-1}{T}_\beta{}^\nu = \delta^\nu_\mu.$$

PROOF. We have by virtue of (3.6) and (4.1)

$$T_\mu{}^\beta\,\overset{-1}{T}_\beta{}^\nu = g_{\lambda\mu}{}^*g^{\beta\lambda}g_{\beta\gamma}{}^*g^{\gamma\nu} = g_{\lambda\mu}\delta^\lambda_\gamma{}^*g^{\gamma\nu} = g_{\lambda\mu}{}^*g^{\lambda\nu} = \delta^\nu_\mu.$$

This equation shows that the determinants of $T_\mu{}^\nu$ and $\overset{-1}{T}_\mu{}^\nu$ are different from zero, *i.e.*, the rank of these tensors is equal to 4.

THEOREM (4.2). *The tensors $T_\mu{}^\nu$ and $\overset{-1}{T}_\mu{}^\nu$ are different from δ^ν_μ. In particular,*

(4.3) $$T_\mu{}^\nu - \delta^\nu_\mu = 2({}^*k^\nu{}_\mu + k_{\lambda\mu}{}^*k^{\nu\lambda}).$$

PROOF. Using (3.6) and (4.1) we obtain

(4.4) $$T_\mu{}^\nu - \delta^\nu_\mu = g_{\lambda\mu}{}^*g^{\nu\lambda} - g_{\lambda\mu}{}^*g^{\lambda\nu} = 2g_{\lambda\mu}{}^*k^{\nu\lambda}.$$

Assume

$$g_{\lambda\mu} \, {}^*k^{\nu\lambda} = 0. \tag{4.5}$$

Then (1.3)a leads at once to

$${}^*k^{\nu\lambda} = 0,$$

and this equation contradicts (3.14). Hence, (4.5) is at variance with (1.3)a. Therefore, by virtue of (4.4) we have $T_\mu{}^\nu \neq \delta^\nu_\mu$, and according to (4.2), also $\overset{-1}{T}_\mu{}^\nu \neq \delta^\nu_\mu$. Moreover, the equations (4.4) and (1.2)a lead at once to (4.3).

5. Basic vectors: Generalities.

Let $P_\lambda{}^\nu$ be an arbitrary tensor. A vector v^ν which satisfies the equation

$$\rho v^\nu = P_\lambda{}^\nu v^\lambda$$

is called an eigenvector of $P_\lambda{}^\nu$, and ρ is the corresponding eigenvalue of $P_\lambda{}^\nu$.

DEFINITION (5.1). *The eigenvectors of $k_\lambda{}^\nu$ will be termed basic vectors (of X_4), and the corresponding eigenvalues of $k_\lambda{}^\nu$ will be called basic scalars (of X_4).* 1) 2).

The following two Theorems will be proved simultaneously.

THEOREM (5.1)a. *Let a^ν be a basic vector and $\lambda \neq 0$ its corresponding basic scalar:*

$$\lambda a^\nu = k_\mu{}^\nu a^\mu. \tag{5.1)a}$$

Then a^ν is a null vector:

$$h_{\lambda\mu} a^\lambda a^\mu = 0. \tag{5.1)b}$$

THEOREM (5.1)*b*. *Let a^ν, b^ν be two basic vectors and $\lambda, \mu \neq -\lambda$ their corresponding basic scalars. Then a^ν and b^ν are perpendicular:*

$$h_{\lambda\mu} a^\lambda b^\mu = 0. \tag{5.2}$$

PROOF. The tensor $k_{\mu\lambda}$ is skew symmetric. Therefore, we have

---

1 In the case of the third class there are vectors which also are called basic vectors in spite of the fact that they are not eigenvectors of $k_\lambda{}^\nu$ (*Cf.* Section 12). However, unless explicitly stated otherwise, we shall be dealing only with basic vectors which are eigenvectors of $k_\lambda{}^\nu$.

2 For an invariant correlation, *cf.* Tharrats, V., (1).

according to (5.1)a

$$\lambda h_{\alpha\beta}a^\alpha a^\beta = \lambda a^\nu a_\nu = k_\mu{}^\nu a^\mu a_\nu = k_{\mu\nu}a^\mu a^\nu = 0.$$

Hence, if $\lambda \neq 0$, then (5.1)b holds. The vector b^ν being an eigenvector, we have

$$\mu b^\nu = k_\alpha{}^\nu b^\alpha, \tag{5.3}$$

and therefore,

$$\mu b^\nu a_\nu = k_{\alpha\beta}a^\beta b^\alpha = -\, k_{\alpha\beta}a^\alpha b^\beta = -\, k_\alpha{}^\nu a^\alpha b_\nu = -\, \lambda a^\nu b_\nu.$$

This equation is equivalent to

$$(\mu + \lambda)h_{\alpha\beta}a^\alpha b^\beta = 0$$

and leads to (5.2), whenever $\lambda + \mu \neq 0$.

In the third section we defined the tensors ${}^{(p)}k_\lambda{}^\nu$. In a similar manner we may define the tensors ${}^{(p)}T_\lambda{}^\nu$ and ${}^{(p)}\overset{-1}{T}_\lambda{}^\nu$ starting with $T_\lambda{}^\nu$ and $\overset{-1}{T}_\lambda{}^\nu$. Moreover, if we put

$$g_\lambda{}^\nu \overset{\text{def}}{=} g_{\lambda\mu}h^{\mu\nu} \qquad {}^*g_\mu{}^\nu \overset{\text{def}}{=} {}^*g^{\lambda\nu}h_{\lambda\mu},$$

we may construct ${}^{(p)}g_\lambda{}^\nu$ and ${}^{(p)*}g_\lambda{}^\nu$ in a similar way.

All these tensors are involved in the following

THEOREM (5.2). *Every basic vector a^ν is also an eigenvector of each of the tensors*

$$ {}^{(p)}g_\lambda{}^\nu,\ {}^{(p)*}g_\lambda{}^\nu,\ {}^{(p)}k_\lambda{}^\nu,\ {}^{(p)}T_\lambda{}^\nu,\ {}^{(p)}\overset{-1}{T}_\lambda{}^\nu. \tag{5.4}$$

PROOF. First, we have according to (5.1)a

$$a^\lambda\, {}^{(2)}k_\lambda{}^\nu = (a^\lambda k_\lambda{}^\alpha)k_\alpha{}^\nu = \lambda a^\alpha k_\alpha{}^\nu = \lambda^2\, a^\nu,$$

and in a similar manner,

$$a^\lambda\, {}^{(p)}k_\lambda{}^\nu = \lambda^p\, a^\nu, \quad p = 2, 3, \ldots. \tag{5.5}$$

Therefore, a^ν is an eigenvector of ${}^{(p)}k_\lambda{}^\nu$. Moreover,

$$a^\alpha g_\alpha{}^\nu = a^\alpha(\delta_\alpha^\nu + k_\alpha{}^\nu) = (1 + \lambda)a^\nu, \tag{5.6}$$

so that a^ν is also an eigenvector of $g_\lambda{}^\nu$ and, by the same authority, also an eigenvector of ${}^{(p)}g_\lambda{}^\nu$. Multiplying (3.6) by a^μ we obtain

$$\begin{aligned} a^\nu = (a^\mu g_{\lambda\mu}){}^*g^{\lambda\nu} = a^\mu(h_{\lambda\mu} + k_{\lambda\mu}){}^*g^{\lambda\nu} = \\ = (1 - \lambda)a_\lambda\, {}^*g^{\lambda\nu} = (1 - \lambda)a^\lambda\, {}^*g_\lambda{}^\nu. \end{aligned} \tag{5.7}$$

Hence, a^ν is an eigenvector of ${}^*g_\lambda{}^\nu$ and, therefore, also an eigenvector of ${}^{(p)*}g_\lambda{}^\nu$. On the other hand, if we multiply (4.1)b by a^μ, we obtain by virtue of (5.7):

$$\begin{aligned} a^\mu \overset{-1}{T}_\mu{}^\nu &= a^\mu(h_{\mu\lambda} + k_{\mu\lambda})^*g^{\lambda\nu} = \\ &= (1+\lambda)a_\lambda{}^*g^{\lambda\nu} = (1+\lambda)a^\lambda{}^*g_\lambda{}^\nu \\ &= \frac{1+\lambda}{1-\lambda}\, a^\nu.^{1} \end{aligned} \tag{5.8}$$

This equation shows that a^ν is an eigenvector of $\overset{-1}{T}_\mu{}^\nu$ and, consequently, also an eigenvector of ${}^{(p)}\overset{-1}{T}_\mu{}^\nu$.

An easy inspection of (4.2) permits us to rewrite this equation in its equivalent form

$$\overset{-1}{T}_\mu{}^\alpha T_\alpha{}^\nu = \delta^\nu_\mu. \tag{5.9}$$

Multiplying this equation by a^μ and remembering (5.8), we see that a^ν is also an eigenvector of $T_\mu{}^\nu$. Hence, it is also an eigenvector of ${}^{(p)}T_\mu{}^\nu$.

6. Basic vectors: First class.

In this section we shall consider only the case of the first class. To this effect we first introduce the scalar

$$D \overset{\text{def}}{=} K^2 - k \tag{6.1)a}$$

and prove the following

LEMMA (6.1). *The scalar D is always positive*

$$D > 0. \tag{6.1)b}$$

PROOF. According to (1.5)b and (1.6)a the scalar k is negative. Moreover, the square K^2 is non-negative. Hence, (6.1)b holds.

THEOREM (6.1). *The basic scalars are $\underset{1}{\lambda}, \ldots, \underset{4}{\lambda}$, given by*

$$\alpha \overset{\text{def}}{=} \underset{1}{\lambda} = -\underset{2}{\lambda} = i\sqrt{\sqrt{D} + K} \neq 0 \tag{6.2)a}$$

$$\beta \overset{\text{def}}{=} \underset{3}{\lambda} = -\underset{4}{\lambda} = \sqrt{\sqrt{D} - K} \neq 0. \tag{6.2)b}$$

Moreover, $i\alpha$ and β are real.

---

[1] The equation (5.7) shows that $\lambda \neq 1$.

PROOF. Using (3.1) we may rewrite (5.1)a in the equivalent form

$$a^{\mu}a_{\nu\mu}(\lambda) = 0. \tag{6.3}$$

A necessary and sufficient condition for the existence of a non-trivial solution a^{ν} of (6.3) is $\mathfrak{a}(\lambda) = 0$, where $\mathfrak{a}(\lambda)$ is the determinant of $a_{\nu\mu}(\lambda)$. According to Theorem (3.1) this condition is equivalent to

$$\lambda^4 + 2\lambda^2 K + k = 0 \tag{6.4}$$

and admits the solutions $\underset{1}{\lambda}, \ldots, \underset{4}{\lambda}$ given by (6.2). The equations (6.1) show that

$$\sqrt{D} > K,$$

so that $i\alpha$ and β are always real and different from zero.

In the next Theorem we denote by $\underset{j}{a}^{\nu}$ the basic vector, whose corresponding basic scalar is $\underset{j}{\lambda}$, given by (6.2) [1].

THEOREM (6.2). *There are only four basic vectors* $\underset{j}{a}^{\nu}$, *and they have the following properties*:

1) *They are defined up to arbitrary factors of proportionality.*
2) *They are null vectors.*
3) *The basic vectors* $\underset{1}{a}^{\nu}$, $\underset{2}{a}^{\nu}$ *are perpendicular to* $\underset{3}{a}^{\nu}$, $\underset{4}{a}^{\nu}$.
4) *They satisfy the conditions*

$$h_{\lambda\mu}\underset{1}{a}^{\lambda}\underset{2}{a}^{\mu} \neq 0 \tag{6.5a}$$

$$h_{\lambda\mu}\underset{3}{a}^{\lambda}\underset{4}{a}^{\mu} \neq 0. \tag{6.5b}$$

5) *The directions of* $\underset{1}{a}^{\nu}$ *and* $\underset{2}{a}^{\nu}$ *are complex-conjugate. The directions of* $\underset{3}{a}^{\nu}, \underset{4}{a}^{\nu}$ *are real.*

PROOF. Since the basic scalars $\underset{j}{\lambda}$ are all distinct and different from zero, we know from elementary algebra that (5.1)a admits only four eigenvectors $\underset{j}{a}^{\nu}$ which are defined up to arbitrary factors of proportionality. Statement (2) follows from Theorem (5.1)a and from (6.2). Statement (3) follows from Theorem (5.1)b and from (6.2). Statement (4) follows from statements (2) and (3). In order to prove statement (5), let us denote by $M_4(\underset{1}{\lambda})$ the matrix of the tensor

---

[1] The construction of $\underset{j}{a}^{\nu}$ will be given later. (*Cf.* Theorem (10.3)).

$$\underset{1}{\lambda}\delta^{\nu}_{\mu} - k_{\mu}{}^{\nu}.$$

Its rank is three, because there is only one basic vector $\underset{1}{a^{\nu}}$ satisfying the condition (5.1)a for $\lambda = \underset{1}{\lambda}$. Hence, if we suppress an appropriate row in $M_4(\underset{1}{\lambda})$, we obtain a 3×4 matrix $M_3(\underset{1}{\lambda})$ whose four minors of the third degree define the direction of $\underset{1}{a^{\nu}}$, *i.e.*,

$$\underset{1}{a^{I}} : \underset{1}{a^{II}} : \underset{1}{a^{III}} : \underset{1}{a^{IV}}. \tag{6.6}$$

On the other hand, the basic scalars $\underset{1}{\lambda}$ and $\underset{2}{\lambda}$ are complex-conjugate, $\underset{2}{\lambda} = \underset{1}{\bar{\lambda}}$. Hence, if we substitute into $M_3(\underset{1}{\lambda})$ for $\underset{1}{\lambda}$ its complex conjugate $\underset{2}{\lambda}$, we obtain a matrix $M_3(\underset{2}{\lambda})$ complex-conjugate to $M_3(\underset{1}{\lambda})$. Therefore, the direction

$$\underset{2}{a^{I}} : \underset{2}{a^{II}} : \underset{2}{a^{III}} : \underset{2}{a^{IV}} \tag{6.7}$$

defined by the corresponding minors of $M_3(\underset{2}{\lambda})$ is complex-conjugate to the direction (6.6). The remaining part of statement (5) is obvious.

THEOREM (6.3). *The factors of proportionality mentioned in statement* (1) *of Theorem* (6.2) *may be chosen in such a way that*

a) *the basic vectors $\underset{1}{a^{\nu}}$, $\underset{2}{a^{\nu}}$ are complex conjugates, the basic vectors $\underset{3}{a^{\nu}}$, $\underset{4}{a^{\nu}}$ are real,*

b) *the basic vectors satisfy the conditions*

$$h_{\lambda\mu}\underset{1}{a^{\lambda}}\underset{2}{a^{\mu}} = h_{\lambda\mu}\underset{3}{a^{\lambda}}\underset{4}{a^{\mu}} = 1. \tag{6.8}$$

The basic vectors $\underset{i}{a^{\nu}}$ satisfying these conditions are defined up to transformations

$$\begin{aligned} '\underset{1}{a^{\nu}} &= e^{i\varphi}\underset{1}{a^{\nu}}, & '\underset{2}{a^{\nu}} &= e^{-i\varphi}\underset{2}{a^{\nu}} \\ '\underset{3}{a^{\nu}} &= \epsilon' e^{\psi}\underset{3}{a^{\nu}}, & '\underset{4}{a^{\nu}} &= \epsilon' e^{-\psi}\underset{4}{a^{\nu}} \quad (\epsilon' = \pm 1) \end{aligned} \tag{6.9}$$

where φ, ψ are arbitrary real scalars.

PROOF. The statements (a), (b) follow immediately from the previous Theorem. The statement about (6.9) follows from (6.8).

AGREEMENT (6.1). *Throughout this book we shall assume that the basic null vectors satisfy the requirements* (*a*) *and* (*b*) *of Theorem* (6.3).

REMARK I. The vectors $\underset{1}{a}^\nu$, $\underset{2}{a}^\nu$ being complex conjugates, the bivector $i\underset{1}{a}^{[\nu}\underset{2}{a}^{\lambda]}$ is a real one. Moreover, the bivectors $i\underset{1}{a}^{[\nu}\underset{2}{a}^{\lambda]}$, $\underset{3}{a}^{[\nu}\underset{4}{a}^{\lambda]}$, both real, are totally perpendicular [1].

REMARK II. The first two equations (6.9) may be looked upon as equations of rotation in the plane of the bivector $i\underset{1}{a}^{[\nu}\underset{2}{a}^{\lambda]}$. The angle of rotation is φ, and the corresponding "isotropic" vectors of this rotation are $\underset{1}{a}^\nu$ and $\underset{2}{a}^\nu$. The last two equations represent a "Minkowski rotation" in the plane of the bivector $i\underset{3}{a}^{[\nu}\underset{4}{a}^{\lambda]}$. [2] The corresponding "isotropic" vectors are the real vectors $\underset{3}{a}^\nu$, $\underset{4}{a}^\nu$, and the angle of this "rotation" is ψ. All four equations (6.9) represent a two-parametric group of motions, the parameters being φ and ψ.

7. Basic vectors: Second and third class. Let us consider first the case of the second class. Then (1.6)b holds so that we have according to (6.1)a

$$D = K^2 > 0. \tag{7.1}$$

THEOREM (7.1). *The basic scalars are* $\underset{1}{\lambda}, \ldots, \underset{4}{\lambda}$, *given by*

$$\left.\begin{aligned} &(7.2)\text{a} \qquad \alpha = \underset{1}{\lambda} = -\underset{2}{\lambda} = \sqrt{-2K} \\ &(7.2)\text{b} \qquad \beta = \underset{3}{\lambda} = -\underset{4}{\lambda} = 0 \end{aligned}\right\} \textit{for } K > 0$$

or

$$\left.\begin{aligned} &(7.3)\text{a} \qquad \alpha = \underset{1}{\lambda} = -\underset{2}{\lambda} = 0 \\ &(7.3)\text{b} \qquad \beta = \underset{3}{\lambda} = -\underset{4}{\lambda} = \sqrt{-2K} \end{aligned}\right\} \textit{for } K < 0.$$

---

[1] *I.e.*, every linear combination of $\underset{1}{a}^\nu$ and $\underset{2}{a}^\nu$ is perpendicular to every linear combination of $\underset{3}{a}^\nu$ and $\underset{4}{a}^\nu$ (and vice versa).

[2] followed by a reflection, if $\epsilon' = -1$.

In the first case αi *is real. In the second case* β *is real.*

PROOF. The equation (6.4) reduces here to

$$\lambda^2(\lambda^2 + 2K) = 0. \tag{7.4}$$

The corresponding basic scalars are described in (7.2) and (7.3) in accordance with (6.2). The remaining part of the Theorem is obvious.

THEOREM (7.2). *There is a one-parameter set of basic vectors belonging to the basic scalar* $\lambda = 0$. *Among these vectors there are only two distinct basic null vectors. Their directions are real* (*complex conjugates*) *in the case* (7.2) (*in the case* (7.3)).

PROOF. For $\lambda=0$ the equation (5.1)a is equivalent to

$$k_{\mu\nu}a^\mu = 0. \tag{7.5}$$

Hence, the matrix of $k_{\mu\nu}$ cannot have the rank four, for $a^\nu \neq 0$ [1]. Since the tensor $k_{\mu\nu} \neq 0$ is skew symmetric, we know from elementary algebra that its rank can be either four or two. Hence, in our case the rank must be two. Therefore, there are at least two distinct basic vectors a^ν and b^ν corresponding to the basic scalar $\lambda = 0$. Every linear combination

$$A^\lambda = \nu a^\lambda + \mu b^\lambda \tag{7.6}$$

with scalars ν, μ also represents a basic vector corresponding to $\lambda = 0$, *i.e.*,

$$A^\mu k_\mu{}^\nu = 0. \tag{7.7)a}$$

In particular, A^ν is a null vector if and only if $\nu : \mu$ is chosen according to

$$\nu^2 h_{\alpha\beta} a^\alpha a^\beta + 2\nu\mu h_{\alpha\beta} a^\alpha b^\beta + \mu^2 h_{\alpha\beta} b^\alpha b^\beta = 0. \tag{7.7)b}$$

Here we have to anticipate a later result [2] according to which (7.7)b admits two distinct roots $\nu : \mu$. Thus, we have only two basic null vectors, which are distinct and satisfy (7.7)a. The remaining part of the proof will be given with the proof of the next Theorem.

THEOREM (7.3). *There are only four basic vectors* $\underset{j}{a}{}^\nu$ *which satisfy the statements* (1) *to* (5) *of Theorem* (6.2).

---

[1] This follows also from $k=0$.

[2] *Cf.* Theorem (11.3).

PROOF. Start with (7.2). Applying the last part of the proof of Theorem (6.2) to this case we obtain two basic null vectors $\underset{1}{a}^{\nu}$, $\underset{2}{a}^{\nu}$ whose directions are complex-conjugate. Denoting by $\underset{3}{a}^{\nu}$, $\underset{4}{a}^{\nu}$ the basic null vectors mentioned in Theorem (7.2), we have four basic null vectors (statement (2)) which are defined up to arbitrary factors of proportionality (statement (1)). Statements (3) and (4) may be proved here in the same way as in the case of Theorem (6.2). The directions of $\underset{1}{a}^{\nu}$, $\underset{2}{a}^{\nu}$ being complex-conjugate, and these vectors being perpendicular to $\underset{3}{a}^{\nu}$, $\underset{4}{a}^{\nu}$, the directions of the latter vectors must be real. The case (7.3) can be treated in a similar way.

THEOREM (7.4). *The factors of proportionality of the basic null vectors may be chosen in such a way that*

a) *the basic null vectors* $\underset{1}{a}^{\nu}$, $\underset{2}{a}^{\nu}$ *are complex-conjugate, while the remaining basic null vectors are real.*

b) *they satisfy the condition* (6.8). [1]

The basic null vectors satisfying these conditions are defined up to the transformations (6.9).

The proof follows the pattern of the proof of Theorem (6.3).

AGREEMENT (7.1). *Throughout this book we shall assume that the basic null vectors satisfy the requirements* (*a*) *and* (*b*) *of Theorem* (7.4).

REMARK. The bivectors $i\underset{1}{a}^{[\nu}\underset{2}{a}^{\lambda]}$ and $\underset{3}{a}^{[\nu}\underset{4}{a}^{\lambda]}$ are always real and totally perpendicular. A remark similar to Remark II (at the end of Section 6) may also be applied to this case.

Let us now consider the case of the third class. Then we have according to (1.6)c and (6.1)a

$$D = K^2 = 0. \tag{7.8}$$

The following two Theorems will be proved simultaneously.

THEOREM (7.5)a. *In the case of the third class there is only one basic scalar, namely*

$$\underset{1}{\lambda} = \dots = \underset{4}{\lambda} = 0. \tag{7.9}$$

---

[1] We disregard the possibility of interchanging the two basic null vectors which belong to $\lambda = 0$.

THEOREM (7.5)b. *There is a one-parameter set of basic vectors belonging to the basic scalar* $\lambda = 0$. *Among these vectors there is only one basic null vector, and this null vector is a real one.*

PROOF. The equation (6.4), which leads to basic scalars, reduces according to (1.6)c to $\lambda^4 = 0$, so that (7.9) holds. The equation (7.7)b, which leads to basic null vectors corresponding to $\lambda = 0$, will be reconsidered later, where it will be shown [1] that in the case of the third class this equation has two coinciding roots $\nu : \mu$, which, of course, must be real.

REMARK. A detailed study of the third class will be undertaken later on. [2]

8. The nonholonomic frame: First and second class.

In this section we shall consider the first two classes. In both cases we have a set of four linearly independent basic null vectors $\underset{j}{a}^\nu$, so that there is only one set of linearly independent covariant vectors $\overset{i}{a}_\lambda$ satisfying the conditions

$$(8.1)\text{a} \qquad \underset{j}{a}^\nu \overset{j}{a}_\lambda = \delta^\nu_\lambda$$

$$(8.1)\text{b} \qquad \underset{j}{a}^\lambda \overset{i}{a}_\lambda = \delta^i_j.$$

From now on $\underset{j}{a}^\nu$ will be termed basic *contravariant* vectors, while $\overset{i}{a}_\lambda$ will be called basic *covariant* vectors.

Let us put

$$(8.2) \qquad A_j{}^\nu \overset{\text{def}}{=} \underset{j}{a}^\nu, \qquad A_\lambda{}^i \overset{\text{def}}{=} \overset{i}{a}_\lambda.$$

The functions $A_j{}^\nu$, $A_\lambda{}^i$ may be looked upon as intermediate components of the unit tensor δ^ν_λ of X_4.

DEFINITION (8.1). *The intermediate components* $A_i{}^\nu$, $A_\lambda{}^i$ *define a nonholonomic frame of* X_4: *If* $T^{\nu\cdots}_{\lambda\cdots}$ *are holonomic components of a tensor, then its nonholonomic components are defined by*

$$(8.3)\text{a} \qquad T^{i\cdots}_{j\cdots} \overset{\text{def}}{=} A_\nu{}^i A_j{}^\lambda \dots T^{\nu\cdots}_{\lambda\cdots}\,{}^{3}$$

---

[1] *Cf.* Theorem (11.4).

[2] *Cf.* Sections 12 and 13.

[3] *Cf.* Scherrer, W., (1), (2).

REMARK. An easy inspection shows that

$$T_{\lambda\cdots}^{\nu\cdots} = A_i{}^{\nu} A_{\lambda\cdots}^{j\cdots} T_{j\cdots}^{i\cdots}. \tag{8.3)b}$$

THEOREM (8.1). *The nonholonomic components*

$$h_{ij} = A_i{}^{\lambda} A_j{}^{\mu} h_{\lambda\mu}, \qquad h^{ij} = A_\lambda{}^i A_\mu{}^j h^{\lambda\mu} \tag{8.4}$$

are given by the matrix equation

$$((h_{ij})) = ((h^{ij})) = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right). \tag{8.5}$$

PROOF. The matrix of h_{ij} expresses the properties of contravariant basic vectors described by Theorems (6.2), (6.3), (7.3) and (7.4). The components h^{ij} are obtained from

$$\begin{aligned} h_{ij}h^{kj} &= A_i{}^{\mu} A_j{}^{\lambda} h_{\mu\lambda} A_\nu{}^k A_\omega{}^j h^{\nu\omega} \\ &= A_i{}^{\mu} A_\nu{}^k h_{\mu\lambda} h^{\nu\lambda} = A_i{}^{\mu} A_\nu{}^k \delta_\mu^\nu = \delta_i^k. \end{aligned} \tag{8.6}$$

THEOREM (8.2). *Denote by* $\mathfrak{A}$ *and* $\mathfrak{a}$ *the determinants of* $A_\lambda{}^i$ *and* $A_i{}^\nu$. *Then*

$$\mathfrak{A}\mathfrak{a} = 1, \tag{8.7}$$

and

$$\left.\begin{aligned} \mathfrak{A} &= \gamma\sqrt{\mathfrak{h}} && \text{(8.8)a} \\ \mathfrak{a} &= \gamma/\sqrt{\mathfrak{h}} && \text{(8.8)b} \end{aligned}\right\} \quad (\gamma \overset{\text{def}}{=} \operatorname{sgn} \mathfrak{A}).^{1}$$

PROOF. We have first

$$\begin{aligned} \mathfrak{a} &= \mathfrak{e}_{\omega\mu\lambda\nu} A_1{}^{\omega} A_2{}^{\mu} A_3{}^{\lambda} A_4{}^{\nu} \\ \mathfrak{A} &= \mathfrak{E}^{\omega\mu\lambda\nu} A_\omega{}^1 A_\mu{}^2 A_\lambda{}^3 A_\nu{}^4. \end{aligned} \tag{8.9}$$

On the other hand, we obtain from (8.4) and (8.5):

$$\begin{aligned} \operatorname{Det}((h_{ij})) &= \mathfrak{a}^2 \mathfrak{h} = 1 \\ \operatorname{Det}((h^{ij})) &= \mathfrak{A}^2 \mathfrak{h}^{-1} = 1, \end{aligned} \tag{8.10}$$

while (8.1) yields (8.7). The equations (8.7) and (8.10) yield (8.8).

THEOREM (8.3). *The nonholonomic components of the indicators are:*

$$\mathfrak{E}^{ijkl} = 4!\, \delta_{[1234]}^{i\,j\,k\,l} \tag{8.11)a}$$

$$\mathfrak{e}_{ijkl} = 4!\, \delta_{i\,j\,k\,l}^{[1234]}, \tag{8.11)b}$$

---

[1] If m is a real number and $M = im$, then we define

$$\operatorname{sgn} M = \operatorname{sgn} m$$

and consequently the nonholonomic components of the pseudotensors $E^{\omega\mu\lambda\nu}$ *and* $e_{\omega\mu\lambda\nu}$ *are:*

(8.12)a $$E^{ijkl} = 4!\,\delta^{ijkl}_{[1234]}$$

(8.12)b $$e_{ijkl} = 4!\,\delta^{[1234]}_{ijkl}.$$

PROOF. Let y^i be a coordinate system for which we have *at a given point P*

$$\frac{\partial y^i}{\partial x^\lambda} = A_\lambda{}^i, \qquad \frac{\partial x^\nu}{\partial y^j} = A_j{}^\nu.$$

Then according to (2.1) (where we replace $x^{\nu'}$ by y^i) we have *at P*

(8.13) $$\mathfrak{E}^{ijkl} = \mathfrak{A}^{-1}\mathfrak{E}^{\omega\mu\lambda\nu} A_\omega{}^i A_\mu{}^j A_\lambda{}^k A_\nu{}^l.$$

On the other hand $\mathfrak{E}^{ijkl}$ as defined by (8.13) are nonholonomic components of $\mathfrak{E}^{\omega\mu\lambda\nu}$, and (8.13) is equivalent with (8.11)a. The equation (8.11)b may be proved in a similar way. The equations (8.12) follow from (2.11), rewritten in the nonholonomic frame, and from (8.10).

In the next Theorem we express the components $A_j{}^\nu$ in terms of $A_\lambda{}^i$ and vice versa:

THEOREM (8.4). *We have*

(8.14)a $$A_i{}^\nu = A_\lambda{}^j h_{ji} h^{\lambda\nu}$$

(8.14)b $$A_\lambda{}^j = A_i{}^\nu h^{ij} h_{\lambda\nu}.$$

Moreover, if i j k l is an even permutation of 1 2 3 4, *we have:*

(8.15)a $$A_i{}^\omega = \gamma\mathfrak{E}^{\omega\mu\lambda\nu} A_\mu{}^j A_\lambda{}^k A_\nu{}^l/\sqrt{\mathfrak{h}}$$

(8.15)b $$A_\omega{}^j = \gamma\mathfrak{e}_{\omega\mu\lambda\nu} A_j{}^\mu A_k{}^\lambda A_l{}^\nu \sqrt{\mathfrak{h}}.$$

PROOF. Put

$$X_i{}^\nu \stackrel{\text{def}}{=} A_\lambda{}^j h_{ji} h^{\lambda\nu}$$

and multiply this equation by $A_\mu{}^i$ remembering (1.4). Then

$$X_i{}^\nu A_\mu{}^i = A_\lambda{}^j A_\mu{}^i h_{ji} h^{\lambda\nu} = h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu.$$

Comparing this equation with (8.1) and (8.2) we obtain $X_i{}^\nu = A_i{}^\nu$. The equation (8.14)b may be obtained in a similar way. If $i\ j\ k\ l$ is an even permutation of 1 2 3 4, then

$$\mathfrak{a} = \mathfrak{e}_{\omega\mu\lambda\nu} A_i{}^\omega A_j{}^\mu A_k{}^\lambda A_l{}^\nu = \gamma/\sqrt{\mathfrak{h}}$$

and

$$A_\omega{}^i = \frac{\partial \ln \mathfrak{a}}{\partial A_i{}^\omega} = \gamma\sqrt{\mathfrak{h}}\, e_{\omega\mu\lambda\nu} A_j{}^\mu A_k{}^\lambda A_l{}^\nu.$$

The equation (8.15)a may be obtained in a similar manner.

REMARK. The equations (8.14) are equivalent to

(8.16)a $$\underset{1}{a^\nu} = \overset{2}{a^\nu}, \quad \underset{2}{a^\nu} = \overset{1}{a^\nu}, \quad \underset{3}{a^\nu} = \overset{4}{a^\nu}, \quad \underset{4}{a^\nu} = \overset{3}{a^\nu}$$

(8.16)b $$\overset{1}{a_\lambda} = \underset{2}{a_\lambda}, \quad \overset{2}{a_\lambda} = \underset{1}{a_\lambda}, \quad \overset{3}{a_\lambda} = \underset{4}{a_\lambda}, \quad \overset{4}{a_\lambda} = \underset{3}{a_\lambda}.$$

9. Continuation. All considerations in this section deal only with the first two classes.

THEOREM (9.1). *The nonholonomic components of* ${}^{(p)}k_\lambda{}^\nu$ *are*

(9.1)a $${}^{(p)}k_x{}^i = \underset{x}{(\lambda)}{}^p \delta^i_x,$$

so that

(9.1)b $${}^{(p)}k_{xi} = \underset{x}{(\lambda)}{}^p h_{xi}.$$

PROOF. Our starting equation is equivalent to (5.5), *i.e.*,

(9.2) $$\underset{x}{(\lambda)}{}^p \underset{x}{a^\nu} = {}^{(p)}k_\mu{}^\nu \underset{x}{a^\mu}.$$

Multiplying this equation by $\overset{i}{a_\nu}$ and remembering (8.1)b, we obtain:

$$\underset{x}{(\lambda)}{}^p \delta^i_x = {}^{(p)}k_\mu{}^\nu \underset{x}{a^\mu} \overset{i}{a_\nu} =$$

$$= {}^{(p)}k_\mu{}^\nu A_x{}^\mu A_\nu{}^i = {}^{(p)}k_x{}^i.$$

Lowering the index i we get (9.1)b.

THEOREM (9.2)a. *The tensors* $h_{\lambda\mu}$, ${}^{(p)}k_{\lambda\mu}$, ${}^{(p)}k^{\lambda\nu}$ *and* ${}^{(p)}k_\lambda{}^\nu$ *may be expressed as follows*:

(9.3)a $$h_{\lambda\mu} = 2\overset{1}{a}_{(\lambda}\overset{2}{a}_{\mu)} + 2\overset{3}{a}_{(\lambda}\overset{4}{a}_{\mu)},$$

(9.3)b $${}^{(p)}k_\lambda{}^\nu = \sum\nolimits_x \underset{x}{(\lambda)}{}^p \overset{x}{a}_\lambda \underset{x}{a^\nu}$$

(9.3)c1 $${}^{(p)}k_{\lambda\mu} = 2\alpha^p \overset{1}{a}_{(\lambda}\overset{2}{a}_{\mu)} + 2\beta^p \overset{3}{a}_{(\lambda}\overset{4}{a}_{\mu)}, \qquad (p \text{ even})$$

$$(9.3)\mathrm{c}_2 \qquad {}^{(p)}k_{\lambda\mu} = 2\alpha^p \overset{1}{a}_{[\lambda} \overset{2}{a}_{\mu]} + 2\beta^p \overset{3}{a}_{[\lambda} \overset{4}{a}_{\mu]}, \qquad (p \text{ odd})$$

$$(9.3)\mathrm{d}_1 \qquad {}^{(p)}k^{\lambda\nu} = 2\alpha^p \underset{2}{a}^{(\lambda} \underset{1}{a}^{\nu)} + 2\beta^p \underset{4}{a}^{(\lambda} \underset{3}{a}^{\nu)}, \qquad (p \text{ even})$$

$$(9.3)\mathrm{d}_2 \qquad {}^{(p)}k^{\lambda\nu} = 2\alpha^p \underset{2}{a}^{[\lambda} \underset{1}{a}^{\nu]} + 2\beta^p \underset{4}{a}^{[\lambda} \underset{3}{a}^{\nu]}, \qquad (p \text{ odd}) .$$

PROOF. The equation (9.3)a is a transcription of

$$(9.4) \qquad h_{\lambda\mu} = A_\lambda{}^i A_\mu{}^j h_{ij}$$

by means of (8.5) and (8.2). In order to prove (9.3)b we start with

$${}^{(p)}k_\lambda{}^\nu = \sum\nolimits_{x,y} {}^{(p)}k_x{}^y A_y{}^\nu A_\lambda{}^x$$

and substitute into this equation from (9.1)a. The remaining equations follow from this and (8.16), (6.2), (7.2) and (7.3).

REMARK. The equations (6.9) and their inverses may be condensed to

$$(9.5)\mathrm{a} \qquad {}'A_\lambda{}^i = C_j{}^i A_\lambda{}^j$$

$$(9.5)\mathrm{b} \qquad {}'A_j{}^\nu = \overset{-1}{C}_j{}^i A_i{}^\nu$$

where

$$(9.6\mathrm{a}) \qquad ((C_j{}^i)) \overset{\text{def}}{=} \left(\begin{pmatrix} e^{-i\varphi} & 0 & 0 & 0 \\ 0 & e^{i\varphi} & 0 & 0 \\ 0 & 0 & \epsilon' e^{-\psi} & 0 \\ 0 & 0 & 0 & \epsilon' e^{\psi} \end{pmatrix}\right)$$

$$(9.6\mathrm{b}) \qquad ((\overset{-1}{C}_j{}^i)) \overset{\text{def}}{=} \left(\begin{pmatrix} e^{i\varphi} & 0 & 0 & 0 \\ 0 & e^{-i\varphi} & 0 & 0 \\ 0 & 0 & \epsilon' e^{\psi} & 0 \\ 0 & 0 & 0 & \epsilon' e^{-\psi} \end{pmatrix}\right).$$

The same equations (9.5) and (9.6) hold for the case of the second class.

The next theorem is an application of Theorem (9.2)a.

THEOREM (9.2)b. *Let v^ν be an arbitrary vector. Then*

$$v^\lambda \, {}^{(4)}k_\lambda{}^\nu + 2Kv^\lambda \, {}^{(2)}k_\lambda{}^\nu = -\, kv^\nu .$$

PROOF. Starting with (9.3)b one obtains

$${}^{(4)}k_\lambda{}^\nu + 2K \, {}^{(2)}k_\lambda{}^\nu = \sum\nolimits^x (\underset{x}{\lambda}^4 + 2K\underset{x}{\lambda}^2)\overset{x}{a}_\lambda \, \underset{x}{a}^\nu .$$

Hence,

$$v^{\lambda}\, {}^{(4)}k_{\lambda}{}^{\nu} + 2Kv^{\lambda}\, {}^{(2)}k_{\lambda}{}^{\nu} + kv^{\nu} = \sum_{x} [\underset{x}{\lambda}^4 + 2K\underset{x}{\lambda}^2 + k]\overset{x}{a}_{\lambda}\underset{x}{a}^{\nu} v^{\lambda} = 0,$$

for, λ being a basic scalar, we have from (6.4)

$$\underset{x}{\lambda}^4 + 2K\underset{x}{\lambda}^2 + k = 0.$$

THEOREM (9.3). *If the nonholonomic frame is subject to the transformations* (9.5), *the nonholonomic components* $T^{i\cdots}_{j\cdots}$ *of a known tensor* $T^{\nu\cdots}_{\lambda\cdots}$ *undergo the transformation*

$$(9.7) \qquad 'T^{i\cdots}_{j\cdots} = C_k{}^i \overset{-1}{C}{}_{j\cdots}^{l\cdots} T^{k\cdots}_{l\cdots}.$$

In particular, the nonholonomic components of

$$h_{\lambda\mu},\ {}^{(p)}k_{\lambda\mu},\ \mathfrak{E}^{\omega\mu\lambda\nu},\ \mathfrak{e}_{\omega\mu\lambda\nu},\ E^{\omega\mu\lambda\nu},\ e_{\omega\mu\lambda\nu}$$

are invariant with respect to (9.5).

PROOF. The equation (9.7) follows from (8.3)a, from

$$'T^{i\cdots}_{j\cdots} = {}'A_{\nu}{}^{i}\, {}'A_{j}{}^{\lambda\cdots}_{\cdots} T^{\nu\cdots}_{\lambda\cdots}$$

and from (9.5). In particular, we have

$$'h_{ij} = \overset{-1}{C}_i{}^k \overset{-1}{C}_j{}^l h_{kl}.$$

Using (9.6) and (8.5) one obtains from this equation

$$(9.8) \qquad 'h_{ij} = h_{ij}.$$

The invariance of the nonholonomic components of ${}^{(p)}k_{\lambda\mu}$ may be proved in a similar way, by means of (9.1). The equations (8.11) and (8.12) show that $\mathfrak{E}^{ijkl}$, E^{ijkl}, and $\mathfrak{e}_{ijkl}$, e_{ijkl} are invariant with respect to (9.5).

10. Construction of $A_i{}^{\nu}$, $A_{\lambda}{}^{i}$ (*First and second class*).[1] In this section we shall consider only the first two classes, so that according to (6.2), (7.2) and (7.3)

$$(10.1) \qquad \alpha^2 - \beta^2 = -2\sqrt{D} < 0.$$

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[1] *Cf.* Hlavatý, V., (18), (19).

Let us first prove the following

LEMMA (10.1). *We have*

(10.2)a $$\mathfrak{E}^{\omega\mu\lambda\nu}\overset{1}{a}_\lambda \overset{2}{a}_\nu = 2\gamma \underset{3}{a}^{[\omega} \underset{4}{a}^{\mu]}\sqrt{\mathfrak{h}}$$

(10.2)b $$\mathfrak{E}^{\omega\mu\lambda\nu}\overset{3}{a}_\lambda \overset{4}{a}_\nu = 2\gamma \underset{1}{a}^{[\omega} \underset{2}{a}^{\mu]}\sqrt{\mathfrak{h}}$$

(10.2)c $$\mathfrak{e}_{\omega\mu\lambda\nu}\underset{1}{a}^\lambda \underset{2}{a}^\nu = 2\gamma \overset{3}{a}_{[\omega} \overset{4}{a}_{\mu]}/\sqrt{\mathfrak{h}}$$

(10.2)d $$\mathfrak{e}_{\omega\mu\lambda\nu}\underset{3}{a}^\lambda \underset{4}{a}^\nu = 2\gamma \overset{1}{a}_{[\omega} \overset{2}{a}_{\mu]}/\sqrt{\mathfrak{h}}.$$

PROOF. Put

(10.3) $$\mathfrak{e}_{\omega\mu} \overset{\text{def}}{=} \mathfrak{e}_{\omega\mu\lambda\nu}\underset{1}{a}^\lambda \underset{2}{a}^\nu$$

and denote by v^ν an arbitrary linear combination of $\underset{1}{a}^\nu$ and $\underset{2}{a}^\nu$. Then according to (10.3)

$$v^\mu \mathfrak{e}_{\omega\mu} = 0,$$

so that $\mathfrak{e}_{\omega\mu}$ and $\underset{1}{a}_{[\omega}\underset{2}{a}_{\mu]}$ are (totally) perpendicular bivectors. Therefore,

(10.4) $$\mathfrak{e}_{\omega\mu} = 2x \overset{3}{a}_{[\omega} \overset{4}{a}_{\mu]},$$

where x has to be found: Using (8.8)b we first obtain

$$\mathfrak{e}_{\omega\mu}\underset{3}{a}^\omega \underset{4}{a}^\mu = \gamma/\sqrt{\mathfrak{h}} = 2x \overset{3}{a}_{[\omega} \overset{4}{a}_{\mu]} \underset{3}{a}^{[\omega} \underset{4}{a}^{\mu]} = x.$$

Substituting this value of x into (10.4) we obtain (10.2)c. The remaining equations (10.2) may be proved in a similar manner.

In the next Theorem we use the following symbols:

(10.5) $$\begin{aligned} -\epsilon_{12} &= \epsilon_{34} \overset{\text{def}}{=} 1 \\ \underset{1}{\nu} &= -\underset{2}{\nu} \overset{\text{def}}{=} \beta\ (= \underset{3}{\lambda} = -\underset{4}{\lambda}) \\ \underset{3}{\nu} &= -\underset{4}{\nu} \overset{\text{def}}{=} \alpha\ (= \underset{1}{\lambda} = -\underset{2}{\lambda}). \end{aligned}$$

THEOREM (10.1). *Put*

(10.6)a $$\overset{xy}{P}_\lambda{}^\nu \overset{\text{def}}{=} \epsilon_{xy}(\underset{x}{\nu}\,\underset{y}{\nu}\,\delta^\nu_\lambda + {}^{(2)}k_\lambda{}^\nu)/2\sqrt{D}$$

(10.6)b $$\overset{xy}{Q}_{\omega\mu} \overset{\text{def}}{=} \epsilon_{xy}\sqrt{\mathfrak{h}}\,\gamma\Big(\underset{x}{\nu}\,\mathfrak{e}_{\omega\mu\lambda\nu}k^{\lambda\nu} - \frac{2\gamma}{\sqrt{\mathfrak{h}}}\,\underset{y}{\lambda}\,k_{\omega\mu}\Big)/4\sqrt{D}$$

$$(xy = 12 \text{ or } xy = 34).$$

Then

(10.7)a $$\overset{12}{P}_\lambda{}^\nu = \sum_1^2{}^z \overset{z}{a}_\lambda \underset{z}{a}^\nu, \quad \text{b)}\ \overset{34}{P}_\lambda{}^\nu = \sum_3^4{}^z \overset{z}{a}_\lambda \underset{z}{a}^\nu$$

(10.7)c $$\overset{xy}{P}_{\lambda\mu} = 2\overset{x}{a}_{(\lambda}\overset{y}{a}_{\mu)}, \quad \text{d)}\ \overset{xy}{P}{}^{\lambda\mu} = 2\underset{y}{a}{}^{(\lambda}\underset{x}{a}{}^{\mu)}$$

and

(10.8)a $$\overset{xy}{Q}_{\omega\mu} = 2\overset{x}{a}_{[\omega}\overset{y}{a}_{\mu]}$$

(10.8)b $$\overset{12}{Q}_\omega{}^\mu = \overset{1}{a}_\omega\underset{1}{a}^\mu - \overset{2}{a}_\omega\underset{2}{a}^\mu$$

(10.8)c $$\overset{34}{Q}_\omega{}^\mu = \overset{3}{a}_\omega\underset{3}{a}^\mu - \overset{4}{a}_\omega\underset{4}{a}^\mu$$

(10.8)d $$\overset{xy}{Q}{}^{\omega\mu} = 2\underset{y}{a}{}^{[\omega}\underset{x}{a}{}^{\mu]}.$$

PROOF. Substituting into (10.6)a for ${}^{(p)}k_\lambda{}^\nu$ from (9.3)b (for $p = 2$), we obtain by virtue of (8.1)a and (10.1):

$$\overset{xy}{P}_\lambda{}^\nu = \epsilon_{xy}\Big[\sum_1^2{}^z \overset{z}{a}_\lambda \underset{z}{a}^\nu(\underset{x}{\nu}\,\underset{y}{\nu} - \underset{1}{\lambda}\,\underset{2}{\lambda}) + \sum_3^4{}^z \overset{z}{a}_\lambda \underset{z}{a}^\nu(\underset{x}{\nu}\,\underset{y}{\nu} - \underset{3}{\lambda}\,\underset{4}{\lambda})\Big]/2\sqrt{D}.$$

From this, remembering (10.5) and (10.1), we easily derive (10.7)a,b. These latter equations, together with (8.16), yield (10.7)c,d. In order to prove (10.8), we observe first that, according to (9.3)c₂, d₂ for $p = 1$ and because of (10.2)c,d, we have:

(10.9) $$\underset{x}{\nu}\,\mathfrak{e}_{\omega\mu\lambda\nu}k^{\lambda\nu} - \frac{2\gamma}{\sqrt{\mathfrak{h}}}\,\underset{y}{\lambda}\,k_{\omega\mu} =$$

$$= \frac{-2\gamma}{\sqrt{\mathfrak{h}}}\,[2(\alpha\underset{x}{\nu} + \beta\underset{y}{\lambda})\,\overset{3}{a}_{[\omega}\overset{4}{a}_{\mu]} + 2(\alpha\underset{y}{\lambda} + \beta\underset{x}{\nu})\,\overset{1}{a}_{[\omega}\overset{2}{a}_{\mu]}].$$

Using (10.5), (10.6)b and (10.9), one obtains (10.8)a. The remaining equations (10.8) follow from (10.8)a and (8.16).

REMARK. The tensors P and Q are very important in the geometry of X_4. The tensor $\overset{xy}{P}_\lambda{}^\nu$ is the unit tensor of the non-holonomic two-dimensional variety spanned by the tensor field $\overset{xy}{Q}{}^{\lambda\nu}$. Both tensors are expressed in terms of basic vectors; however, *their construction does not require the use of these vectors.* On the contrary, they enable us to construct basic vectors. In order to perform this construction we need the following

THEOREM (10.2). *Put*

$$\begin{aligned}
\underset{1}{E}_\lambda{}^\nu &= \overset{1}{E}_\lambda{}^\nu \overset{\text{def}}{=} (\overset{12}{P}_\lambda{}^\nu + \overset{12}{Q}_\lambda{}^\nu)/2 \\
\underset{2}{E}_\lambda{}^\nu &= \overset{2}{E}_\lambda{}^\nu \overset{\text{def}}{=} (\overset{12}{P}_\lambda{}^\nu - \overset{12}{Q}_\lambda{}^\nu)/2 \\
\underset{3}{E}_\lambda{}^\nu &= \overset{3}{E}_\lambda{}^\nu \overset{\text{def}}{=} (\overset{34}{P}_\lambda{}^\nu + \overset{34}{Q}_\lambda{}^\nu)/2 \\
\underset{4}{E}_\lambda{}^\nu &= \overset{4}{E}_\lambda{}^\nu \overset{\text{def}}{=} (\overset{34}{P}_\lambda{}^\nu - \overset{34}{Q}_\lambda{}^\nu)/2.
\end{aligned} \tag{10.10}$$

Then

$$\underset{x}{E}_\lambda{}^\nu = \overset{x}{E}_\lambda{}^\nu = \overset{x}{a}_\lambda \underset{x}{a}^\nu, \quad x = 1, \ldots, 4. \tag{10.11}$$

The proof follows from (10.7)a,b and (10.8)b,c.

THEOREM (10.3). (*Construction of the basic vectors*). *Let* $\underset{x}{u}^\nu$ $(\overset{x}{u}_\lambda)$ *be four arbitrary vectors such that*

$$\underset{x}{u}^\lambda \underset{x}{E}_\lambda{}^\nu \neq 0 \qquad \overset{x}{u}_\nu \overset{x}{E}_\lambda{}^\nu \neq 0. \tag{10.12}$$

Let $\underset{x}{\rho}$, $\overset{x}{\rho}$ *be arbitrary scalars related by*

$$\begin{gathered}
\underset{z}{u}^\lambda \underset{z}{E}_\lambda{}^\alpha \underset{t}{u}^\mu \underset{t}{E}_\mu{}^\beta h_{\alpha\beta} = 1/\underset{z}{\rho}\,\underset{t}{\rho} \\
\overset{z}{u}_\lambda \overset{z}{E}_\alpha{}^\lambda \overset{t}{u}_\mu \overset{t}{E}_\beta{}^\mu h^{\alpha\beta} = 1/\overset{z}{\rho}\,\overset{t}{\rho} \\
(zt = 12 \text{ or } zt = 34).
\end{gathered} \tag{10.13}$$

Then

$$\underset{x}{\rho}\underset{x}{u}^\lambda \underset{x}{E}_\lambda{}^\nu = \underset{x}{a}^\nu \tag{10.14}$$
$$\overset{x}{\rho}\overset{x}{u}_\nu \overset{x}{E}_\lambda{}^\nu = \overset{x}{a}_\lambda.$$

PROOF. Denoting, as usual, by v^x the nonholonomic components of a vector v^ν, we obtain by virtue of (10.11):

$$\underset{x}{u}^\lambda \underset{x}{E}_\lambda{}^\nu = \underset{x}{u}^x \underset{x}{a}^\nu. \tag{10.15}$$

The condition (10.12) shows that $\underset{x}{u}^x \neq 0$, *so that according to* (10.15) $\underset{x}{u}^\lambda \underset{x}{E}_\lambda{}^\nu$ *is proportional to the basic vector* $\underset{x}{a}^\nu$. Therefore, among all products

$$\underset{z}{u}^\lambda \underset{z}{E}_\lambda{}^\alpha \underset{t}{u}^\mu \underset{t}{E}_\mu{}^\beta h_{\alpha\beta} \tag{10.16}$$
$$(z, t = 1, \ldots, 4)$$

only two are different from zero, namely the products (10.16) for $zt = 12$ and $zt = 34$. Therefore, the first of equations (10.13) is not meaningless, and the vectors $\underset{x}{\rho}\underset{x}{u}^\lambda \underset{x}{E}_\lambda{}^\nu$ satisfy, according to (10.13), the relation

$$\underset{x}{\rho}\underset{x}{u}^\lambda \underset{x}{E}_\lambda{}^\alpha \underset{y}{\rho}\underset{y}{u}^\mu \underset{y}{E}_\mu{}^\beta h_{\alpha\beta} = h_{xy}$$

for $x, y = 1, \ldots, 4$, where h_{xy} is defined by (8.5). *Hence, these vectors are basic vectors.* A similar method may be used to prove the second equation (10.14).

REMARK I. According to (8.2) we could have written in the last three Theorems $A_i{}^\nu$, $A_\lambda{}^j$ instead of $\underset{i}{a}^\nu$, $\overset{j}{a}_\lambda$.

REMARK II. The vectors $\underset{x}{u}^\nu$ ($\overset{x}{u}_\lambda$) involved in the last Theorem need not be linearly independent. Thus, in order to obtain $\underset{1}{a}^\nu$ and $\underset{2}{a}^\nu$ we could have used one vector $\underset{1}{u}^\nu = \underset{2}{u}^\nu$ and so on. In any case, we are able to find the nonholonomic components v^x (w_x) of a vector v^ν (w_λ) without actually computing the basic vectors from (5.1)a:

$$v^x = v^\lambda \overset{x}{u}_\nu \overset{x}{\rho} \overset{x}{E}_\lambda{}^\nu, \qquad w_x = w_\nu \underset{x}{u}^\lambda \underset{x}{\rho} \underset{x}{E}_\lambda{}^\nu. \tag{10.17}$$

REMARK III. For our later purposes Theorem (10.2) is much more important than the last Theorem. In the second Chapter we shall deal with situations where Theorem (10.2) will be very often needed. We shall exhibit at least one example of such a situation here.

EXAMPLE (10.1). Let K_λ be a given vector and $\underset{x}{f}$ a set of given scalars. A new vector S_λ is defined by means of its nonholonomic components

$$(10.18) \qquad S_x = \underset{x}{f} K_x.$$

One has to find the components S_λ. From (10.18) we obtain

$$(10.19) \qquad S_\lambda = \sum{}^x S_x A_\lambda{}^x = \sum{}^x \underset{x}{f} K_x A_\lambda{}^x = \sum{}^x \underset{x}{f} \underset{x}{E}_\lambda{}^\nu K_\nu.$$

Therefore, if we put

$$(10.20) \qquad 2Y_\lambda{}^\nu \overset{\text{def}}{=} \{\delta_\lambda^\nu[-\alpha^2(\underset{3}{f} + \underset{4}{f}) + \beta^2(\underset{1}{f} + \underset{2}{f})]$$
$$- {}^{(2)}k_\lambda{}^\nu[\underset{1}{f} + \underset{2}{f} - \underset{3}{f} - \underset{4}{f}]\}/2\sqrt{D}$$
$$+ \sqrt{\mathfrak{h}}\gamma\Big\{ \mathfrak{e}_{\omega\mu\lambda\alpha} k^{\omega\mu} h^{\alpha\nu}[-\beta(\underset{1}{f} - \underset{2}{f}) + \alpha(\underset{3}{f} - \underset{4}{f})]$$
$$+ \frac{2\gamma}{\sqrt{\mathfrak{h}}} k_\lambda{}^\nu[-\alpha(\underset{1}{f} - \underset{2}{f}) + \beta(\underset{3}{f} - \underset{4}{f})]\Big\}/4\sqrt{D},$$

we have by virtue of (10.19), (10.10) and (10.6)

$$(10.21) \qquad S_\lambda = Y_\lambda{}^\nu K_\nu.$$

11. Digression on line geometry.[1]

Consider a point O of X_4 and the following transformations at O:

$$(11.1) \qquad \begin{gathered} v^{\nu'} = c_\lambda{}^{\nu'} v^\lambda, \qquad w_{\lambda'} = c_{\lambda'}{}^\nu w_\nu \\ \left(c_\lambda{}^{\nu'} \overset{\text{def}}{=} \left(\frac{\partial x^{\nu'}}{\partial x^\lambda}\right)_O \quad c_{\lambda'}{}^\nu \overset{\text{def}}{=} \left(\frac{\partial x^\nu}{\partial x^{\lambda'}}\right)_O\right). \end{gathered}$$

The underlying space for the group of centro-affine transformations

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[1] In this section we assume a knowledge of the elements of line geometry. The reader not familiar with this topic will find a sufficient introduction to it in Appendix III.

(11.1) is the linear tangential four-dimensional space T_4 of X_4 at O. Denote by P_3 the ideal three-dimensional space of T_4. It is a projective space, where v^ν are homogeneous projective coordinates of a point and w_λ are homogeneous projective coordinates of a plane. Then (11.1) may be looked upon as a group of projective transformations in P_3.

The space P_3 is endowed with a quadric H whose projective homogeneous plane (point) coordinates are $h_{\lambda\mu}(h^{\lambda\mu})$. The relationship between a point v^ν and its polar plane v_λ with respect to H is given by

$$(11.2) \qquad v^\lambda = h^{\lambda\nu} v_\nu, \qquad v_\mu = h_{\mu\lambda} v^\lambda.$$

Therefore, *these equations must not be regarded as defining the raising or lowering of indices in* P_3! The operation of raising or lowering indices in P_3 will be performed by means of $E^{\omega\mu\lambda\nu}$ or $e_{\omega\mu\lambda\nu}$ and will apply to skew symmetric quadratic tensors only. Thus, for instance, if $a^{\lambda\mu}$, $b_{\lambda\mu}$ are two such tensors, we define $\underline{a}_{\lambda\mu}$ and $\underline{b}^{\lambda\mu}$ as follows:

$$(11.3)\text{a} \qquad \underline{a}_{\lambda\mu} = \tfrac{1}{2} e_{\lambda\mu\omega\nu} a^{\omega\nu}, \qquad \underline{b}^{\lambda\mu} = \tfrac{1}{2} E^{\lambda\mu\omega\nu} b_{\omega\nu}.$$ [1]

Let us stress that

$$(11.3)\text{b} \qquad \underline{\underline{a}}^{\lambda\mu} = \tfrac{1}{2} E^{\lambda\mu\omega\nu} \underline{a}_{\omega\nu} = \tfrac{1}{4} E^{\lambda\mu\omega\nu} e_{\omega\nu\alpha\beta} a^{\alpha\beta} = a^{\lambda\mu},$$

and in a similar manner

$$(11.3)\text{c} \qquad \underline{\underline{b}}_{\lambda\mu} = b_{\lambda\mu}.$$

Every skew symmetric quadratic tensor defines *a linear line complex* (for short, a complex) in P_3. This also holds for the tensor $k_{\omega\mu}$: The corresponding complex is a three-parameter set of lines whose Plücker point coordinates $p^{\lambda\nu}$ are related by [1]

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[1] The following explanatory remarks have to be added:

(1) In Appendix III we raised or lowered the indices by means of $\mathfrak{E}^{\omega\mu\nu\lambda}$ and $\mathfrak{e}_{\omega\mu\lambda\nu}$. Because the coordinate transformations were restricted to projective ones with the determinant equal to 1, the indicators became tensors which we denoted by $I^{\omega\mu\lambda\nu}$ and $i_{\omega\mu\lambda\nu}$. However, here we do not restrict our transformations (11.1) and have at our disposal the tensor $h_{\lambda\mu}$. Therefore, we may use the pseudotensors E and e instead of I and i. Since the components of tensors a and b are homogeneous projective coordinates in P_3, we may multiply them by an arbitrary factor without changing the geometrical significance of a and b in P_3.

(2) We have to underline the kernel letter here whenever we raise or lower the indices in P_3 by means of E and e. This is necessary in order to avoid confusion with the operations described by (11.2).

$$k_{\lambda\mu}p^{\lambda\mu} = 0. \tag{11.4}$$

Besides the complex $k_{\mu\lambda}$ we also have a complex $k^{\lambda\nu}$ whose plane coordinates are

$$\underline{k}_{\omega\mu} = \tfrac{1}{2}e_{\omega\mu\lambda\nu}k^{\lambda\nu}.$$

The main scope of this section is the investigation of the relationship of the complexes $k_{\omega\mu}$ and $k^{\lambda\nu}$.

THEOREM (11.1). *The complexes* $k_{\lambda\mu}$ *and* $k^{\lambda\nu}$ *are polar conjugates with respect to* H: *To every ruling* $p^{\lambda\nu}$ *of* $k_{\lambda\nu}$ *there corresponds a ruling* $p_{\lambda\mu}$ *of* $k^{\lambda\nu}$, *polar conjugate with respect to* H *and vice versa.*

PROOF. Consider a ruling $p^{\lambda\nu}$ of $k_{\lambda\nu}$, so that (11.4) is satisfied. The ruling $p^{\lambda\nu}$ is defined by any two of its arbitrary distinct points v^ν and w^ν

$$p^{\nu\lambda} = 2v^{[\nu}w^{\lambda]}.$$

Its polar-conjugate line is the intersection of the polar planes

$$v_\lambda = h_{\lambda\nu}v^\nu, \qquad w_\lambda = h_{\lambda\nu}w^\nu,$$

i.e.,

$$\begin{aligned} p_{\lambda\mu} &= 2v_{[\lambda}w_{\mu]} = 2h_{\lambda\alpha}h_{\mu\beta}v^{[\alpha}w^{\beta]} \\ &= h_{\lambda\alpha}h_{\mu\beta}p^{\alpha\beta}. \end{aligned} \tag{11.5}$$

On the other hand, (11.4) is equivalent (in X_4) to

$$k^{\lambda\nu}p_{\lambda\nu} = 0, \tag{11.6}$$

so that $p_{\lambda\nu}$ given by (11.5) is a ruling of $k^{\lambda\nu}$ in P_3. On the other hand, if we start with (11.6), we obtain (11.4) for

$$p^{\lambda\nu} = p_{\alpha\beta}h^{\alpha\lambda}h^{\beta\nu}.$$

Let us now investigate the geometrical significance of

$$k = 0 \tag{11.7a}$$

$$K = 0. \tag{11.7b}$$ [2]

---

[1] We know that a necessary and sufficient condition for $p^{\lambda\nu}$ to be a line is

$$e_{\omega\mu\lambda\nu}p^{\omega\mu}p^{\lambda\nu} = 0,$$

and this equation is equivalent to

$$p_{\omega\mu}p^{\omega\mu} = 0.$$

[2] *Cf.* (1.5)b,c.

THEOREM (11.2). *A necessary and sufficient condition for the complex* $k_{\omega\mu}$ $(k^{\lambda\nu})$ *to be a special one is: The tensor* $k_{\omega\mu}$ *does not belong to the first class. If this condition is satisfied, then* $k_{\omega\mu}$ $(k^{\lambda\nu})$ *is the axis of the complex* $k_{\omega\mu}$ $(k^{\lambda\nu})$.

PROOF. A necessary and sufficient condition for the complex $k_{\omega\mu}$ to be a special one is

$$(11.7)\text{c} \qquad E^{\omega\mu\lambda\nu}k_{\omega\mu}k_{\lambda\nu} = 0,$$

and this equation is equivalent to (11.7)a (*Cf.* (2.10) and (1.5)b.) On the other hand, (11.7)c is a necessary and sufficient condition for $k_{\omega\mu}$ to be a line. Hence, if (11.7)c is satisfied, the complex $k_{\omega\mu}$ is a special one, and $k_{\omega\mu}$ are also the coordinates of its axis. The proof for $k^{\lambda\nu}$ follows from

$$E^{\omega\mu\lambda\nu}\underline{k}_{\omega\mu}\underline{k}_{\lambda\nu} = e_{\alpha\beta\gamma\delta}k^{\alpha\beta}k^{\gamma\delta} = -E^{\alpha\beta\gamma\delta}k_{\alpha\beta}k_{\gamma\delta}$$

and from (11.7)c.
The following two Theorems will be proved simultaneously:

THEOREM (11.3). *If* $k_{\omega\mu}$ *belongs to the second class, then the axes* $k_{\omega\mu}$, $k^{\lambda\nu}$ *of the special complexes are polar conjugates with respect to H and are skew. Therefore, each of them intersects H in two distinct points. In particular, if* $K > 0$, *then the axes* $k_{\omega\mu}$, $k^{\lambda\nu}$ *are given by the Plücker coordinates*

$$\underset{1}{a}_{[\omega}\underset{2}{a}_{\mu]} \textit{ and } \underset{1}{a}^{[\omega}\underset{2}{a}^{\mu]},^{1}$$

while for $K < 0$ *these axes are given by the Plücker coordinates*

$$\underset{3}{a}_{[\omega}\underset{4}{a}_{\mu]} \textit{ and } \underset{3}{a}^{[\omega}\underset{4}{a}^{\mu]}.^{1}$$

---

[1] $\underset{i}{a}_{\omega}$ is the polar plane (with respect to H) of the point $\underset{i}{a}^{\omega}$. This latter point being a point of H, the plane $\underset{i}{a}_{\omega}$ is the tangential plane of H at $\underset{i}{a}^{\omega}$. Hence, $\underset{1}{a}_{[\omega}\underset{2}{a}_{\mu]}$ $(\underset{3}{a}_{[\omega}\underset{4}{a}_{\mu]})$ is the intersection line of the tangential planes $\underset{1}{a}_{\omega}, \underset{2}{a}_{\mu}$, $(\underset{3}{a}_{\omega}, \underset{4}{a}_{\mu})$, and as such, it is polar-conjugate with respect to H to the line $\underset{1}{a}^{[\omega}\underset{2}{a}^{\mu]}$ $(\underset{3}{a}^{[\omega}\underset{4}{a}^{\mu]})$. In other words, $\underset{1}{a}_{[\omega}\underset{2}{a}_{\mu]}$ $(\underset{3}{a}_{[\omega}\underset{4}{a}_{\mu]})$ is identical with the line $\underset{3}{a}^{[\omega}\underset{4}{a}^{\mu]}$ $(\underset{1}{a}^{[\omega}\underset{2}{a}^{\mu]})$. This is the geometrical interpretation in P_3 of lowering and raising of indices of skew symmetric quadratic tensors by means of $e_{\omega\mu\lambda\nu}$ and $E^{\omega\mu\lambda\nu}$.

THEOREM (11.4). *If* $k_{\omega\mu}$ *belongs to the third class, then the axes* $k_{\omega\mu}$, $k^{\lambda\nu}$ *of the special complexes are polar-conjugate tangential lines of* H. *Therefore, they intersect* H *in their common point.*

PROOF. The polar-conjugacy of $k_{\omega\mu}$ and $k^{\lambda\nu}$ is obvious. A necessary and sufficient condition that these two lines meet is

$$0 = E^{\omega\mu\lambda\nu} k_{\omega\mu} \underline{k}_{\lambda\nu} = 2k_{\alpha\beta}k^{\alpha\beta} = 8K.$$

Hence, in the case of the second class these polar-conjugate lines are skew, and therefore, each of them intersects H in two distinct points.[1] The remaining part of Theorem (11.3) follows from (9.3)c$_2$, (9.3)d$_2$ (for $p = 1$) and from (7.2) and (7.3). If, however, the tensor $k_{\omega\mu}$ is of the third class, then (11.7)a,b hold, and the polar-conjugate lines $k_{\omega\mu}$ and $k^{\lambda\nu}$ meet. Therefore, they have to be incident with a tangential plane of H. The signature of H being $(+ + + -)$, this quadric is a non-ruled one and non-degenerate. Consequently, neither one of the lines $k_{\omega\mu}$, $k^{\lambda\nu}$ could be a ruling of H. Therefore, they are tangential lines of H and meet in a common point of H.

THEOREM (11.5). *In the case of the first class the complexes* $k_{\omega\mu}$ *and* $k^{\lambda\nu}$ *are general ones and intersect in a hyperbolic congruence, whose axes are*

$$\underset{1}{a}^{[\omega}\underset{2}{a}^{\mu]} \text{ and } \underset{3}{a}^{[\omega}\underset{4}{a}^{\mu]}. \tag{11.8}$$

PROOF. The generality of the complex $k_{\omega\mu}$ ($k^{\lambda\nu}$) follows from Theorem (11.2). Moreover, we may use the equations (9.3)c$_2$ and (10.2)a,b to find the homogeneous coordinates $\underline{k}^{\lambda\nu}$ to be

$$\alpha\underset{3}{a}^{[\lambda}\underset{4}{a}^{\nu]} + \beta\underset{1}{a}^{[\lambda}\underset{2}{a}^{\nu]}, \tag{11.9}$$

while (9.3)d$_2$ shows that the homogeneous coordinates $k^{\lambda\nu}$ may be written

$$\alpha\underset{1}{a}^{[\lambda}\underset{2}{a}^{\nu]} + \beta\underset{3}{a}^{[\lambda}\underset{4}{a}^{\nu]}. \tag{11.10}$$

Hence, both complexes belong to a one-parameter set of complexes

$$A\underset{1}{a}^{[\lambda}\underset{2}{a}^{\nu]} + B\underset{3}{a}^{[\lambda}\underset{4}{a}^{\nu]} \tag{11.11}$$

which all intersect in the congruence mentioned in the Theorem.

---

[1] *Cf.* (7.7)b.

We already know the geometrical significance of the equations (11.7). In the next Theorem we shall find the geometrical role played by the scalars k, K.

THEOREM (11.6). *Let $k_{\omega\mu}$ be of the first class. Then the projective angle Φ of the complexes $k_{\lambda\mu}$ and $k^{\lambda\nu}$ is given by the following equation:*

$$\Phi = \ln \frac{\alpha}{\beta}, \tag{11.12}$$

where α, β are defined by (6.2).

PROOF. Every complex in the set (11.11) is defined by its projective coordinate $A : B$. In particular, this coordinate is

$$\begin{array}{l} \beta : \alpha \text{ for } \underline{k}^{\lambda\nu} \\ \alpha : \beta \text{ for } k^{\lambda\nu} \\ 1 : 0 \text{ for } \underset{1}{a}{}^{[\lambda} \underset{2}{a}{}^{\nu]} \\ 0 : 1 \text{ for } \underset{3}{a}{}^{[\lambda} \underset{4}{a}{}^{\nu]}. \end{array} \tag{11.13}$$

The projective angle Φ of $k_{\omega\mu}$ and $k^{\lambda\nu}$ (or, which is the same, of $\underline{k}^{\lambda\nu}$ and $k^{\lambda\nu}$) is

$$\Phi = \tfrac{1}{2} \ln \Gamma, \tag{11.14}$$

where Γ is the cross ratio of the four complexes (11.13) (*i.e.*, the cross ratio of their coordinates $A : B$). A simple inspection shows $\Gamma = \dfrac{\alpha^2}{\beta^2}$, so that the equation (11.14) reduces to (11.12).[1]

REMARK I. We know that two complexes intersecting in a hyperbolic congruence are projectively orthogonal[2] if their projective angle is equal to $\pm i\pi/2$. It may be easily shown that this happens for the angle (11.12) if and only if $K = 0$. *This is the geometrical interpretation of* (11.7)b *for the case when $k_{\omega\mu}$ is of the first class.*

REMARK II. Theorem (11.5) is a starting point for spinor algebra and spinor analysis of $g_{\lambda\mu}$. (*Cf.* Appendix II.)

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[1] The geometrical interpretation of the projective angle of two complexes is given in Appendix III.

[2] Sometimes they are called complexes in involution. *Cf.* Debever, R., (1).

12. Nonholonomic frame: Third class. The considerations of the previous section enable us to introduce a nonholonomic frame even in the case of the third class. We shall first operate in the space P_3 and shall use the following abbreviations: The line $k_{\omega\mu}$ will be denoted by L; its polar-conjugate line (with respect to H) will be denoted by L^*. These two lines are tangential lines of H and meet in a point of H (*cf.* Theorem (11.4))[1]. Moreover, we denote by M an arbitrary line in the plane (L, L^*), not incident with the point (L, L^*), and we denote by M^* its polar-conjugate line. The line M^* is obviously incident with the point (L, L^*) (because its polar conjugate is incident with the plane (L, L^*)).

THEOREM (12.1). *There is a two parameter set consisting of four linearly independent real points* $\underset{1}{a}^\nu, \ldots, \underset{4}{a}^\nu$ *which have the following properties*:

a) $\underset{4}{a}^\nu$ *is the point on* H *where* L, L^* *and* M^* *meet.* $\underset{3}{a}^\nu \neq \underset{4}{a}^\nu$ *is the intersection point* (L, M), $\underset{2}{a}^\nu$ *is the intersection point* (L^*, M), *and finally,* $\underset{1}{a}^\nu \neq \underset{4}{a}^\nu$ *is the other intersection point of* M^* *and* H.

b) *These points satisfy the relations*:

$$\underset{1}{a}^\lambda k_\lambda{}^\nu = \underset{2}{a}^\nu \tag{12.1)a}$$

$$\underset{2}{a}^\lambda k_\lambda{}^\nu = \underset{4}{a}^\nu \tag{12.1)b}$$

$$\underset{f}{a}^\lambda k_\lambda{}^\nu = 0 \quad (f = 3,4). \tag{12.1)c}$$

c) *If we put*

$$h_{ij} \overset{\text{def}}{=} h_{\lambda\mu} \underset{i}{a}^\lambda \underset{j}{a}^\mu, \tag{12.2}$$

then

$$((h_{ij})) = \left(\left(\begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right)\right)^{2}. \tag{12.3}$$

PROOF. The existence of points $\underset{3}{a}^\lambda$, $\underset{4}{a}^\lambda$ which satisfy (12.1)c and $h_{44} = 0$, $h_{33} \neq 0$ is assured by Theorems (11.4) and (7.5)b. The factors of proportionality of these points being arbitrary, we choose the factor of $\underset{3}{a}^{\nu}$ in such a way that $h_{33} = +1$[1]. The Plücker plane coordinates of L are $k_{\omega\mu}$, and because L^* is polar-conjugate to L, $k_{\omega\mu}$ is proportional to $\underset{4}{a}_{[\omega}\underset{2}{a}_{\mu]}$, *i.e.*, $k_{\omega\mu}$ is the intersection of the polar planes $\underset{4}{a}_{\omega}$, $\underset{2}{a}_{\omega}$ of the points $\underset{4}{a}^{\omega}$, $\underset{2}{a}^{\omega}$. Let us choose the factor of proportionality of $\underset{4}{a}^{\nu}$ in such a way that

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[1] See also Cahen, M., (1).

[2] Note that (12.3) holds for the *third* class, while (8.5) refers to the first two classes.

$$-\,k_{\omega\mu} = 2\underset{4}{a}_{[\omega}\underset{2}{a}_{\mu]}. \tag{12.4}$$

Then we have

$$-\,\underset{2}{a}^{\omega}k_{\omega\mu} = (\underset{2}{a}^{\omega}\underset{4}{a}_{\omega})\underset{2}{a}_{\mu} - (\underset{2}{a}^{\omega}\underset{2}{a}_{\omega})\underset{4}{a}_{\mu}. \tag{12.5}$$

The point $\underset{2}{a}^{\omega}$ is incident with L^*, which is incident with $\underset{4}{a}_{\omega}$. Therefore, if we use the fact that $\underset{2}{a}^{\nu}$ is outside of H, we may choose its factor of proportionality in such a way that $h_{22} = +1$, and consequently, (12.5) reduces to

$$\underset{2}{a}^{\omega}k_{\omega\mu} = \underset{4}{a}_{\mu}$$

which is equivalent to (12.1)b. Moreover, multiplying (12.4) by $\underset{1}{a}^{\omega}$ and remembering that $\underset{1}{a}^{\nu}$ is incident with M^* which is the intersection of $\underset{2}{a}_{\omega}$ and $\underset{3}{a}_{\omega}$, we obtain

$$\begin{aligned} -\,\underset{1}{a}^{\omega}k_{\omega\mu} &= (\underset{1}{a}^{\omega}\underset{4}{a}_{\omega})\underset{2}{a}_{\mu} - (\underset{1}{a}^{\omega}\underset{2}{a}_{\omega})\underset{4}{a}_{\mu} \\ &= (\underset{1}{a}^{\omega}\underset{4}{a}_{\omega})\underset{2}{a}_{\mu}. \end{aligned} \tag{12.6}$$

The point $\underset{1}{a}^{\omega}$ is on H, *i.e.*, $h_{11} = 0$, so that we are still free to choose its factor of proportionality. If we choose it according to $h_{14} = -1$, the equation (12.6) reduces to

$$\underset{1}{a}^{\omega}k_{\omega\mu} = \underset{2}{a}_{\mu}, \tag{12.7}$$

---

[1] The point $\underset{3}{a}^{\nu}$ is outside H; therefore, h_{33} cannot be negative.

equivalent to (12.1)a. The factors of all points $\underset{i}{a}^{\nu}$ are now fixed, and the remaining numbers h_{ij} result from the location of these points on polar-conjugate lines L, L^* and M, M^*. The only arbitrary line of these four lines is the line M in $\underset{4}{a}_{\omega}$. Therefore, the set $\underset{i}{a}^{\nu}$ described in a) is a two-parameter set.

REMARK. The factors of proportionality of the points $\underset{i}{a}^{\nu}$ being fixed, we may interpret $\underset{i}{a}^{\nu}$ in X_4 as components of four linearly independent contravariant vectors: $\underset{1}{a}^{\nu}$ and $\underset{4}{a}^{\nu}$ are null vectors, and $\underset{2}{a}^{\nu}$ and $\underset{3}{a}^{\nu}$ are unit vectors. Although only $\underset{4}{a}^{\nu}$ and $\underset{3}{a}^{\nu}$ are eigenvectors of $k_{\lambda}{}^{\nu}$, *we shall call all four vectors basic vectors, keeping in mind that* $\underset{2}{a}^{\nu}$ *and* $\underset{1}{a}^{\nu}$ *are not eigenvectors of* $k_{\lambda}{}^{\nu}$. The vectors $\underset{i}{a}^{\nu}$ being linearly independent, we may define the corresponding set of covariant vectors $\overset{i}{a}_{\lambda}$ by means of (8.1). Then, of course, we may use (8.2) as a definition of the nonholonomic frame and apply Definition (8.1). In particular, h_{ij} defined by (12.2) and satisfying (12.3) are nonholonomic components of $h_{\lambda\mu}$. The nonholonomic components h^{ij} of $h^{\lambda\mu}$ are defined by

(12.8) $$h^{ij} = A_{\lambda}{}^{i} A_{\mu}{}^{j} h^{\lambda\mu}$$

and satisfy the condition

(12.9) $$((h_{ij})) = ((h^{ij})),$$ [1]

which will be used later.

THEOREM (12.2). *Denote by* $\mathfrak{A}$ *and* $\mathfrak{a}$ *the determinants of* $A_{\lambda}{}^{i}$ *and* $A_{i}{}^{\nu}$. *Then*

(12.10)a $$\mathfrak{A}\mathfrak{a} = 1,$$

and

(12.10)b $$\mathfrak{A} = \gamma\sqrt{|\mathfrak{h}|}, \qquad \mathfrak{a} = \gamma/\sqrt{|\mathfrak{h}|}, \gamma = \operatorname{sgn} \mathfrak{A}.$$ [2]

The proof based on (12.2), (12.3), (12.8) and (12.9) follows the pattern of the proof of Theorem (8.2).

---

[1] This condition follows easily from

$$h_{ij}h^{ik} = \delta_j^k.$$

[2] $\mathfrak{A}$ and $\mathfrak{a}$ are real for the third class.

THEOREM (12.3). *The components* $A_i{}^\nu$, $A_\lambda{}^i$ *satisfy* (8.14). *Moreover, if ijkl is an even permutation of* 1 2 3 4, *we have*

$$A_i{}^\omega = \gamma \mathfrak{E}^{\omega\mu\lambda\nu} A_\mu{}^j A_\lambda{}^k A_\nu{}^l / \sqrt{|\mathfrak{h}|} \tag{12.11a}$$

$$A_\omega{}^i = \gamma \mathfrak{e}_{\omega\mu\lambda\nu} A_j{}^\mu A_k{}^\lambda A_l{}^\nu \sqrt{|\mathfrak{h}|}. \tag{12.11b}$$

The proof is similar to the proof of Theorem (8.4).

REMARK. The relations (8.14) are here equivalent to

$$\begin{aligned} &\underset{1}{a}^\nu = -\overset{4}{a}^\nu,\ \underset{2}{a}^\nu = \overset{2}{a}^\nu,\ \underset{3}{a}^\nu = \overset{3}{a}^\nu,\ \underset{4}{a}^\nu = -\overset{1}{a}^\nu \\ &\overset{1}{a}_\lambda = -\underset{4}{a}_\lambda,\ \overset{2}{a}_\lambda = \underset{2}{a}_\lambda,\ \overset{3}{a}_\lambda = \underset{3}{a}_\lambda,\ \overset{4}{a}_\lambda = -\underset{1}{a}_\lambda. \end{aligned} \tag{12.12}$$

13. Continuation.

The considerations of this section concern only the third class.

THEOREM (13.1). *The nonholonomic components* $k_i{}^j$, k_{ij} *and* k^{ij} *are*

$$k_1{}^2 = \quad k_2{}^4 = 1, \quad \textit{the remaining } k_i{}^j = 0 \tag{13.1a}$$

$$k_{12} = -k_{21} = 1, \quad \textit{the remaining } k_{ij} = 0 \tag{13.1b}$$

$$k^{42} = -k^{24} = -1, \quad \textit{the remaining } k^{ij} = 0. \tag{13.1c}$$

The nonholonomic components ${}^{(p)}k_i{}^j$, ${}^{(p)}k_{ij}$, *and* ${}^{(p)}k^{ij}$ *are*

$${}^{(p)}k_1{}^4 = \delta^p_2 \quad \textit{the remaining } {}^{(p)}k_i{}^j = 0 \tag{13.2a}$$

$${}^{(p)}k_{11} = -\delta^p_2 \ \textit{the remaining } {}^{(p)}k_{ij} = 0 \tag{13.2b}$$

$${}^{(p)}k^{44} = -\delta^p_2 \ \textit{the remaining } {}^{(p)}k^{ij} = 0. \tag{13.2c}$$

$$p = 2, 3, \ldots.$$

The proof of the first half, based on (12.1) and (12.3), is similar to the proof of Theorem (9.1). The proof of (13.2) follows from (12.3), (13.1) and (3.8).

THEOREM (13.2). *The tensors* $h_{\lambda\mu}$, ${}^{(p)}k_{\lambda\mu}$, ${}^{(p)}k^{\lambda\nu}$ *and* ${}^{(p)}k_\lambda{}^\nu$ *may be expressed as follows:*

$$h_{\lambda\mu} = -2\overset{1}{a}_{(\lambda}\overset{4}{a}_{\mu)} + \overset{2}{a}_\lambda \overset{2}{a}_\mu + \overset{3}{a}_\lambda \overset{3}{a}_\mu \tag{13.3a}$$

$${}^{(p)}k_\lambda{}^\nu = \delta^p_1(\overset{1}{a}_\lambda \underset{2}{a}^\nu + \overset{2}{a}_\lambda \underset{4}{a}^\nu) + \delta^p_2 \overset{1}{a}_\lambda \underset{4}{a}^\nu \ {}^{1} \tag{13.3b}$$

$$p = 1, 2, \ldots.$$

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[1] See also Kilmister, C., (1).

$$\begin{aligned}
&(13.3)\text{c} \quad {}^{(p)}k_{\lambda\mu} = -\,2\delta^p_1 \overset{2}{a}_{[\lambda} \overset{1}{a}_{\mu]} - \delta^p_2 \overset{1}{a}_\lambda \overset{1}{a}_\mu \\
&(13.3)\text{d} \quad {}^{(p)}k^{\lambda\nu} = -\,2\delta^p_1 \underset{4}{a}^{[\lambda} \underset{2}{a}^{\nu]} - \delta^p_2 \underset{4}{a}^\lambda \underset{4}{a}^\nu.
\end{aligned} \qquad p = 1, 2, \ldots .$$

PROOF. We have

$${}^{(p)}k_\lambda{}^\nu = {}^{(p)}k_i{}^j \overset{i}{a}_\lambda \underset{j}{a}^\nu.$$

Substituting into this equation from (13.1)a and (13.2)a we obtain (13.3)b. The remaining formulae may be proved in a similar way.

THEOREM (13.3). *Let* $\underset{j}{a}^\nu$ *and* $'\underset{j}{a}^\nu$ *be two sets described in Theorem* (12.1). *Then the corresponding nonholonomic frames* $A_i{}^\nu$, $A_\lambda{}^i$, *and* $'A_i{}^\nu$, $'A_\lambda{}^i$ *are related by the formulae*

$$(13.4) \qquad \text{a)} \quad 'A_\lambda{}^i = C_j{}^i A_\lambda{}^j \qquad \text{b)} \quad 'A_j{}^\nu = \overset{-1}{C}{}_j{}^i A_i{}^\nu$$

with

$$(13.5)\text{a}^1 \quad ((\overset{-1}{C}{}_j{}^i)) = \left(\begin{pmatrix} \epsilon & u\cos v & u\sin v & \dfrac{\epsilon}{2}u^2 \\ 0 & \epsilon & 0 & u\cos v \\ 0 & 0 & \epsilon' & \epsilon\epsilon' u\sin v \\ 0 & 0 & 0 & \epsilon \end{pmatrix}\right)$$

and

$$(13.5)\text{b} \quad ((C_j{}^i)) = \left(\begin{pmatrix} \epsilon & -\,u\cos v & -\,\epsilon\epsilon' u\sin v & \dfrac{\epsilon u^2}{2} \\ 0 & \epsilon & 0 & -\,u\cos v \\ 0 & 0 & \epsilon' & -\,u\sin v \\ 0 & 0 & 0 & \epsilon \end{pmatrix}\right),$$

where $\epsilon, \epsilon' = \pm 1$ *and* u, v *are two arbitrary parameters.*

PROOF. According to the description of the set $\underset{j}{a}^\nu$ in Theorem (12.1), any other set $'\underset{j}{a}^\nu$ satisfying the conditions of this Theorem

---

[1] In (13.5)a,b the subscripts denote the rows; the superscripts indicate the columns.

is related to $\underset{i}{a}^\nu$ by means of the following equations:

$$
\begin{aligned}
'\underset{1}{a}^\nu &= A\underset{1}{a}^\nu + B\underset{2}{a}^\nu + C\underset{3}{a}^\nu + D\underset{4}{a}^\nu \\
'\underset{2}{a}^\nu &= \phantom{A\underset{1}{a}^\nu +{}} E\underset{2}{a}^\nu \phantom{{}+ C\underset{3}{a}^\nu} + F\underset{4}{a}^\nu \\
'\underset{3}{a}^\nu &= \phantom{A\underset{1}{a}^\nu + B\underset{2}{a}^\nu +{}} G\underset{3}{a}^\nu + H\underset{4}{a}^\nu \\
'\underset{4}{a}^\nu &= \phantom{A\underset{1}{a}^\nu + B\underset{2}{a}^\nu + C\underset{3}{a}^\nu +{}} I\underset{4}{a}^\nu,
\end{aligned}
\tag{13.6}
$$

where the coefficients $A, \ldots, I$ may be obtained from the invariance of h_{ij}, k_{ij} with respect to (13.6).[1] In this way we obtain (13.5)a, while (13.5)b follows from (13.5)a and

$$C_k{}^j \overset{-1}{C}_j{}^i = \delta^i_k.$$

REMARK I. It is easy to see that the nonholonomic components $T^{i\cdots}_{j\cdots}$ of a known tensor transform under (13.4) according to

$$'T^{i\cdots}_{j\cdots} = C_k{}^i \overset{-1}{C}_j{}^{l\cdots}_{\cdots} T^{k\cdots}_{l\cdots}. \tag{13.7}$$

In particular, h_{ij}, h^{ij}, and k_{ij} (as well as nonholonomic components built up from these components) are invariant with respect to (13.4) and (13.5).

REMARK II. It is interesting to compare Theorem (9.3) and the statement of the previous remark. In both cases we have a two-parameter group of transformations. But there the analogy stops. The difference may best be described in terms of the geometry of the space P_3. In the case of the first two classes the group of transformations leaves the tetrahedron $\underset{1}{a}^\nu \underset{2}{a}^\nu \underset{3}{a}^\nu \underset{4}{a}^\nu$ invariant. In the case of the third class only a point (*i.e.*, $\underset{4}{a}^\nu$), its polar plane (*i.e.*, $\underset{4}{a}_\lambda$), and two lines L and L^*, incident with $\underset{4}{a}_\lambda$ and $\underset{4}{a}^\nu$, are left invariant.

The theory of nonholonomic frames as developed here will be the most useful tool in dealing with the problem B (*Cf.* the next chapter).

14. The tensor $^*h^{\lambda\nu}$.

In this very short section we shall consider an application of the theory of nonholonomic frames. The result will be needed later.

---

[1] This invariance is evident from (12.3) and (13.1)b.

THEOREM (14.1). *The tensor* ${}^*h^{\lambda\nu}$ *as defined by* (3.9)a *is of rank four.*

PROOF. Assume first that $k_{\lambda\mu}$ is not of the third class. Then, according to (9.1)a and (10.5), we may write for the nonholonomic components ${}^*h_j{}^i$ of ${}^*h_\lambda{}^\nu = {}^*h^{\alpha\nu}h_{\alpha\lambda}$

$$g\,{}^*h_x{}^j = \delta^j_x(1+2K) + \delta^j_x \underset{x}{\lambda}{}^2 =$$
$$= \delta^j_x(1-\underset{x}{\nu}{}^2).$$

On the other hand, by virtue of (1.3)a,

$$(1-\underset{1}{\nu}{}^2)\,(1-\underset{2}{\nu}{}^2)\,(1-\underset{3}{\nu}{}^2)\,(1-\underset{4}{\nu}{}^2) =$$
$$= [(1-\alpha^2)\,(1-\beta^2)]^2 = g^2 \neq 0.$$

Hence, the factor $1-\underset{x}{\nu}{}^2$ is always different from zero, so that the rank of ${}^*h_x{}^j$ is equal to four. By the same token, the rank of ${}^*h^{\lambda\nu}$ is equal to four.

Assume now that $k_{\lambda\mu}$ belongs to the third class. Then we have according to (3.9)a and (13.2)a

$$(({}^*h_i{}^j)) = \left(\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right).$$

Hence, even in the case of the third class the rank of ${}^*h_i{}^j$ and, therefore, the rank of ${}^*h^{\lambda\nu}$ is equal to four.

15. The tensor $f_{\lambda\mu}$ [1]. In this section we shall investigate the tensor field which will later be identified with the electromagnetic field [2]. In the definition of this tensor we shall use the quantity σ defined in the introductory section.

THEOREM (15.1). *Put*

$$f_{\lambda\nu} \overset{\text{def}}{=} -\frac{\sigma}{2}\,\mathfrak{e}_{\lambda\nu\alpha\beta}\,{}^*k^{\alpha\beta}\sqrt{|\mathfrak{g}|}. \tag{15.1}$$

---

[1] *Cf.* Hlavatý, V., (13).

[2] *Cf.* Hlavatý, V., (13), (15), (17), (20); Todeschini, B., (1); and Ikeda, H., (2).

Then $f_{\lambda\nu}$ *is a tensor, and*

(15.2)a $$f_{\lambda\nu} = \frac{\sigma}{\sqrt{g}}\,[\kappa\sqrt{|k|}\,k_{\lambda\nu} - \tfrac{1}{2}e_{\lambda\nu\alpha\beta}k^{\alpha\beta}]$$

(15.2)b $$f^{\lambda\nu} = \frac{\sigma}{\sqrt{g}}\,[\kappa\sqrt{|k|}\,k^{\lambda\nu} + \tfrac{1}{2}E^{\lambda\nu\alpha\beta}k_{\alpha\beta}].$$

PROOF. Multiplying the pseudodensity $\sqrt{|\mathfrak{g}|}$ by σ we obtain a density (of weight $+ 1$). Therefore, $f_{\lambda\nu}$, as defined by (15.1), is a tensor. Substituting from (3.9)b into (15.1) and using (2.2)c and (1.3)b, we obtain

$$\begin{aligned} f_{\lambda\nu} &= -\frac{\sigma}{2\mathfrak{g}}\,\mathfrak{e}_{\lambda\nu\alpha\beta}\sqrt{|\mathfrak{g}|}\left[\mathfrak{h}k^{\alpha\beta} + \frac{\kappa}{2}\sqrt{\mathfrak{k}}\,\mathfrak{E}^{\omega\mu\alpha\beta}k_{\omega\mu}\right] \\ &= \frac{\sigma}{2\sqrt{|\mathfrak{g}|}}\,[\mathfrak{h}k^{\alpha\beta}\mathfrak{e}_{\lambda\nu\alpha\beta} + 2\kappa\sqrt{\mathfrak{k}}\,k_{\lambda\nu}] \\ &= \frac{\sigma}{\sqrt{g}}\,[\kappa\sqrt{|k|}\,k_{\lambda\nu} - \tfrac{1}{2}e_{\lambda\nu\alpha\beta}k^{\alpha\beta}]. \end{aligned}$$

The equation (15.2)b follows easily from (15.2)a and (2.13)b.

Let us denote by $\mathfrak{f}$ the determinant of $f_{\lambda\nu}$, and by f the scalar

(15.3) $$f \overset{\text{def}}{=} \mathfrak{f}/\mathfrak{h}.$$

THEOREM (15.2). *We have*:

(15.4) $$\text{a) } \mathfrak{f} = \mathfrak{k}, \qquad \text{b) } f = k.$$

PROOF. Using the method of the proof of Theorem (2.3) we first obtain

(15.5) $$64\mathfrak{f} = (\mathfrak{E}^{\omega\mu\lambda\nu}f_{\omega\mu}f_{\lambda\nu})^2.$$

On the other hand, we get from (15.2)a

(15.6) $$\begin{aligned} g\mathfrak{E}^{\omega\mu\lambda\nu}f_{\omega\mu}f_{\lambda\nu} &= |k|\,\mathfrak{E}^{\omega\mu\lambda\nu}k_{\omega\mu}k_{\lambda\nu} \\ &\quad -\kappa\sqrt{|k\mathfrak{h}|}\,\mathfrak{E}^{\omega\mu\lambda\nu}\mathfrak{e}_{\lambda\nu\alpha\beta}k_{\omega\mu}k^{\alpha\beta} + \tfrac{1}{4}\mathfrak{E}^{\omega\mu\lambda\nu}|\mathfrak{h}|\,\mathfrak{e}_{\omega\mu\alpha\beta}\mathfrak{e}_{\lambda\nu\gamma\delta}k^{\alpha\beta}k^{\gamma\delta} = \\ &= -\,8\sqrt{\mathfrak{k}}\,\kappa[1 + 2K + k] = -\,8\kappa g\sqrt{\mathfrak{k}}. \end{aligned}$$

Comparing (15.5) and (15.6), we obtain (15.4)a. The equation (15.4)b follows from (15.4)a and (1.5)b.

THEOREM (15.3). *The equation* (15.2)a *admits a unique inverse* *solution*:

$$k_{\lambda\nu} = \frac{\sigma\sqrt{g}}{1-f}\,[\tfrac{1}{2}e_{\lambda\nu\alpha\beta}f^{\alpha\beta} - \varphi\sqrt{|f|}\,f_{\lambda\nu}] \tag{15.7}$$

$$(\varphi \overset{\text{def}}{=} \operatorname{sgn} \mathfrak{E}^{\omega\mu\lambda\nu} f_{\omega\mu} f_{\lambda\nu}).$$ [1]

PROOF. Assume that $k_{\lambda\nu}$ belongs to one of the last two classes. Then $k = 0$, and according to (15.4) we also have $f = 0$. Then (15.7) is obviously the inverse solution of (15.2)a. Assume now that $k_{\lambda\mu}$ belongs to the first class, so that $fk \neq 0$. Then we see from (9.3) and (15.2) that $f_{\lambda\mu}$ is a linear combination of $\overset{1}{a}_{[\lambda}\overset{2}{a}_{\mu]}$ and $\overset{3}{a}_{[\lambda}\overset{4}{a}_{\mu]}$. Therefore we may put

$$k_{\lambda\nu} = a e_{\lambda\nu\alpha\beta}f^{\alpha\beta} + b f_{\lambda\nu}, \tag{15.8}$$

where a and b are to be found. Substituting from (15.8) into (15.2) and using (2.12), (2.13), we see that by (15.4)b the resulting equation is an identity if and only if

$$a = \frac{\sigma}{2}\,\frac{\sqrt{g}}{1-f}$$

$$b = -\,\frac{\sigma\sqrt{g}}{1-f}\,\varphi\sqrt{|f|}.$$

Substituting these values into (15.8) we obtain (15.7).

In the first section (Definition (1.1)) we defined the class of $k_{\lambda\mu}$ by means of k and K. We could adopt the same definition of the class of $f_{\lambda\mu}$, using f and

$$4F \overset{\text{def}}{=} f_{\alpha\beta}f^{\alpha\beta}.$$

Then the following Theorem holds.

THEOREM (15.4). *The tensors $k_{\lambda\mu}$ and $f_{\lambda\mu}$ are of the same class. Moreover, they have the same set of basic null eigenvectors.* [2, 3]

---

[1] Since $f \leq 0$ we always have $f \neq 1$. Moreover, from (15.6) we see that $\varphi = -\varkappa$.

[2] According to our definition of basic vectors in the case of the third class (*Cf*. Remark following the proof of Theorem (12.1)) there is only one basic null vector, namely $\underset{4}{a}^{\nu}$, in this case.

[3] *Cf*. also Sáenz, W., (1).

PROOF. First we obtain from (15.7)

$$4K = -\frac{g}{(1-f)^2}[4(1+f)F + 8f]. \tag{15.9)a}$$

Assume first $k \neq 0$. Then we also have $f \neq 0$, and consequently, both tensors belong to the first class. If, however, $k = 0$, then we also have $f = 0$, and (15.9)a reduces to

$$K = -gF. \tag{15.9)b}$$

Hence, if $k_{\lambda\mu}$ belongs to the second (third) class, the same holds for $f_{\lambda\mu}$ and vice versa. Moreover, the equation (15.2)a reduces in this case to

$$f_{\lambda\nu} = -\frac{\sigma}{2\sqrt{g}} e_{\lambda\nu\alpha\beta}k^{\alpha\beta}. \tag{15.10}$$

Assume now that $k_{\lambda\mu}$ is of the third class. Then we have according to (13.3)d

$$\underset{4}{a}^{\lambda} f_{\lambda\nu} = \frac{\sigma}{\sqrt{g}} e_{\lambda\nu\alpha\beta}\underset{4}{a}^{\alpha}\underset{2}{a}^{\beta}\underset{4}{a}^{\lambda} = 0. \tag{15.11}$$

Hence, $\underset{4}{a}^{\nu}$ is also the null eigenvector of $f_{\lambda\nu}$, and since $f_{\lambda\nu}$ is of the third class, it is the unique null eigenvector of $f_{\lambda\nu}$. Assume now that $k_{\lambda\mu}$ is of the second class and that $K > 0$. Then we have according to (7.2), (9.3)d$_2$ and (15.10)

$$\begin{aligned} \underset{q}{a}^{\lambda} f_{\lambda\nu} &= -\frac{\sigma\alpha}{\sqrt{g}} e_{\lambda\nu\alpha\beta}\underset{2}{a}^{\alpha}\underset{1}{a}^{\beta}\underset{q}{a}^{\lambda} = 0, \quad q = 1, 2. \\ \underset{h}{a}^{\lambda} f_{\lambda\nu} &= -\frac{\sigma\alpha}{\sqrt{g}} e_{\lambda\nu\alpha\beta}\underset{2}{a}^{\alpha}\underset{1}{a}^{\beta}\underset{h}{a}^{\lambda}, \quad h = 3, 4. \end{aligned} \tag{15.12}$$

We know from Theorem (8.4) that

$$e_{\lambda\nu\alpha\beta}\underset{2}{a}^{\alpha}\underset{1}{a}^{\beta}\underset{h}{a}^{\lambda}$$

is proportional to $\overset{h'}{a}_{\nu} = \underset{h}{a}_{\nu}$, $(h, h' = 3, 4;\ h' \neq h)$. Hence, according to (15.12), the basic null vectors of $k_{\lambda\mu}$ are also null eigenvectors of $f_{\lambda\nu}$. The same method used in the case $K < 0$ leads to the same result (the pairs $\underset{1}{a}^{\nu}$, $\underset{2}{a}^{\nu}$ and $\underset{3}{a}^{\nu}$, $\underset{4}{a}^{\nu}$ interchanging their roles).

Assume now that $k_{\lambda\nu}$ is of the first class. Then we see from (9.3)d$_2$ that

$$\frac{\sigma}{2} e_{\lambda\nu\alpha\beta} k^{\alpha\beta} \underset{u}{a}^{\lambda} = \sigma[\alpha e_{\lambda\nu\alpha\beta} \underset{2}{a}^{\alpha} \underset{1}{a}^{\beta} \underset{u}{a}^{\lambda} + \beta e_{\lambda\nu\alpha\beta} \underset{4}{a}^{\alpha} \underset{3}{a}^{\beta} \underset{u}{a}^{\lambda}]. \tag*{(15.13)}$$

Hence, according to Theorem (8.4) the vector (15.13) is proportional to $\overset{u'}{a}_{\nu}$, where $u' \neq u$ and $u, u' = 1, 2$ or $u, u' = 3, 4$. In other words, the left hand term in (15.13) is proportional to $\underset{u}{a}_{\nu}$. Therefore, we see from (15.2) that $\underset{u}{a}^{\lambda}$ is also a null eigenvector of $f_{\lambda\nu}$. This tensor being of class one we have only four corresponding null eigenvectors. Hence, both tensors $k_{\lambda\mu}$ and $f_{\lambda\mu}$ also have the same set of null eigenvectors in the case of the first class.

In the next Theorem we shall use the scalar

$$e = 1 + 2F + f. \tag*{(15.14)}$$

THEOREM (15.5). *If*

$$1 + f \neq 0,\text{[1]} \tag*{(15.15)a}$$

then $e > 0$, *and*

$$\frac{\sqrt{g}}{1 - f} = \frac{1}{\sqrt{e}}, \tag*{(15.16)}$$

so that (15.7) *can be written*

$$k_{\lambda\nu} = \frac{\sigma}{\sqrt{e}} [\tfrac{1}{2} e_{\lambda\nu\alpha\beta} f^{\alpha\beta} - \varphi \sqrt{|f|} f_{\lambda\nu}]. \tag*{(15.17)}$$

PROOF. Substituting from (15.9)a and (15.4)b into (3.3)b, we obtain:

$$g \left\{ 1 + \frac{1}{2(1 - f)^2} [4F(1 + f) + 8f] \right\} = 1 + f. \tag*{(15.18)}$$

If $1 + f = 0$, then the factor of g in (15.18) is equal to zero. If $f + 1 \neq 0$, then (15.18) yields

$$g = \frac{(1 - f)^2}{e}. \tag*{(15.19)}$$

---

[1] The case $1 + f = 0$ is considered in Hlavatý, V., (13).

This equation shows that $e > 0$. Therefore, we obtain from it (15.16). The remaining part of the Theorem is obvious.

AGREEMENT (15.1). *Throughout this book we shall assume*

$$k + 1 \neq 0. \tag{15.20}$$

NOTE. Remembering (15.4) we obtain from (15.20)

$$f + 1 \neq 0. \tag{15.21}$$

This concludes the necessary algebraic prerequisites for the subsequent investigations.

CHAPTER II

THE CONNECTION $\Gamma_\lambda{}^\nu{}_\mu$

PART I

The case of general n

Introduction. [1] [2] A set of functions $\Gamma_\lambda{}^\nu{}_\mu$ with the transformation rule

$$(1) \qquad \Gamma_{\lambda'}{}^{\nu'}{}_{\mu'} = \frac{\partial x^{\nu'}}{\partial x^{\alpha}}\left(\frac{\partial x^{\beta}}{\partial x^{\lambda'}}\frac{\partial x^{\gamma}}{\partial x^{\mu'}}\Gamma_\beta{}^\alpha{}_\gamma + \frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\partial x^{\mu'}}\right)$$

is termed *a connection*, and $\Gamma_\lambda{}^\nu{}_\mu$ are its *coefficients*. Thus, for instance, if $\Gamma_\lambda{}^\nu{}_\mu$ are coefficients of a connection, the same holds for

$$(2) \qquad \Lambda_\lambda{}^\nu{}_\mu \overset{\text{def}}{=} \Gamma_{(\lambda}{}^\nu{}_{\mu)} \quad (= \tfrac{1}{2}(\Gamma_\lambda{}^\nu{}_\mu + \Gamma_\mu{}^\nu{}_\lambda)),$$

while

$$(3) \qquad S_{\lambda\mu}{}^\nu \overset{\text{def}}{=} \Gamma_{[\lambda}{}^\nu{}_{\mu]} \quad (= \tfrac{1}{2}(\Gamma_\lambda{}^\nu{}_\mu - \Gamma_\mu{}^\nu{}_\lambda))$$

is a tensor, called the *torsion tensor* of the connection $\Gamma_\lambda{}^\nu{}_\mu$. If $S_{\lambda\mu}{}^\nu = 0$, the connection is called a *symmetric* one.

If v^ν (w_λ) is a vector, then

$$(4) \qquad \begin{gathered} D_\mu v^\nu \overset{\text{def}}{=} \partial_\mu v^\nu + \Gamma_\lambda{}^\nu{}_\mu v^\lambda \\ D_\mu w_\lambda \overset{\text{def}}{=} \partial_\mu w_\lambda - \Gamma_\lambda{}^\nu{}_\mu w_\nu \\ \left(\partial_\mu \overset{\text{def}}{=} \frac{\partial}{\partial x^\mu}\right) \end{gathered}$$

is called the *covariant derivative* of v^ν (w_λ). D_μ is a symbolic covariant vector, called the *symbol of the covariant derivative* with respect to $\Gamma_\lambda{}^\nu{}_\mu$. The application of D_μ to more general tensors follows the pattern of (4). Thus, for instance,

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[1] In this part of the second chapter we shall deal with the case of general $n > 1$. In other words, the restriction $n = 4$ is dropped. Of course, all the results are also valid for $n = 4$.

[2] The results of this part are taken from Hlavatý, V., (7).

$$D_\omega g_{\lambda\mu} = \partial_\omega g_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \Gamma_\mu{}^\alpha{}_\omega g_{\lambda\alpha}. \tag{5}$$

If $\mathfrak{e}$ is a density of weight w, then

$$D_\mu \mathfrak{e} = \partial_\mu \mathfrak{e} - w\Gamma_\alpha{}^\alpha{}_\mu \mathfrak{e} \tag{6}$$

is its covariant derivative. $D_\mu \mathfrak{e}$ is a covariant vector density of weight w. [1]

The formulae for the covariant derivatives of pseudotensors or pseudodensities are the same as for tensors and densities.

A particular case of a symmetric connection is the connection whose coefficients are the so-called *Christoffel symbols*

$$\{\lambda{}^\nu{}_\mu\} \overset{\text{def}}{=} \tfrac{1}{2}h^{\nu\alpha}(\partial_\lambda h_{\alpha\mu} + \partial_\mu h_{\alpha\lambda} - \partial_\alpha h_{\lambda\mu}). \tag{7}$$

The covariant derivative with respect to $\{\lambda{}^\nu{}_\mu\}$ will be denoted ∇_μ. The connection $\{\lambda{}^\nu{}_\mu\}$ has the property

$$\nabla_\omega h_{\lambda\mu} \overset{\text{def}}{=} \partial_\omega h_{\lambda\mu} - \{\lambda{}^\alpha{}_\omega\} h_{\alpha\mu} - \{\mu{}^\alpha{}_\omega\} h_{\lambda\alpha} = 0. \tag{8}$$

As a matter of fact, this equation is used in the gravitational field theory for defining the Christoffel symbols.

1. Analogy to the gravitational field theory. In the unified field theory the role of the basic tensor is no longer played by $h_{\lambda\mu}$, for we have here the non-symmetric tensor $g_{\lambda\mu}$. Therefore, it would seem natural to follow the analogy of the gravitational theory in defining the unified connection $\Gamma_\lambda{}^\nu{}_\mu$ by an equation corresponding to (8) of the Introduction, *i.e.*, by

$$D_\omega g_{\lambda\mu} \overset{\text{def}}{=} \partial_\omega g_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \Gamma_\mu{}^\alpha{}_\omega g_{\lambda\alpha} = 0. \tag{1.1}$$

However, this equation imposes a heavy restriction on $g_{\lambda\mu}$:

THEOREM (1.1). *A necessary condition that* (1.1) *admit a solution* $\Gamma_\lambda{}^\nu{}_\mu$ *is: The scalars* $\mathfrak{g}$ *and* $\mathfrak{k}$, *defined by* I (1.5) [2] *are constant.*

---

[1] The same holds for each of the following expressions:

$$\partial_\mu \mathfrak{e} - w\Gamma_\mu{}^\alpha{}_\alpha \mathfrak{e}$$
$$\partial_\mu \mathfrak{e} - wA_\alpha{}^\alpha{}_\mu \mathfrak{e}.$$

However, if $S_{\lambda\mu}{}^\nu \neq 0$, only (6) is consistent with (4).

[2] Here and in subsequent references we shall use the Roman numerals I, II, III, ..., VI to indicate the first, ..., the sixth Chapter.

PROOF. First one obtains from (1.1)

$$D_{\omega} h_{\lambda\mu} = 0 \tag{1.2)a}$$

$$D_{\omega} k_{\lambda\mu} = 0, \tag{1.2)b}$$

so that, in particular,

$$*g^{\lambda\mu} D_{\omega} g_{\lambda\mu} = \partial_{\omega} \ln \mathfrak{g} - 2\Gamma_{\alpha}{}^{\alpha}{}_{\omega} = 0 \tag{1.3)a}$$

$$h^{\lambda\mu} D_{\omega} h_{\lambda\mu} = \partial_{\omega} \ln \mathfrak{h} - 2\Gamma_{\alpha}{}^{\alpha}{}_{\omega} = 0. \tag{1.3)b}$$

Hence,

$$\partial_{\omega} \ln \mathfrak{g} = \partial_{\omega} \ln \mathfrak{h}.$$

Integrating this equation one gets $g =$ const. If $k = 0$, then our Theorem is proved. If $k \neq 0$, then there is a unique tensor $\hat{k}^{\lambda\mu} = \hat{k}^{[\lambda\mu]}$ such that

$$\hat{k}^{\lambda\alpha} k_{\mu\alpha} = \hat{k}^{\alpha\lambda} k_{\alpha\mu} = \delta^{\lambda}_{\mu},$$

and consequently, (1.2)b gives

$$\hat{k}^{\lambda\mu} D_{\omega} k_{\lambda\mu} = \partial_{\omega} \ln \mathfrak{k} - 2\Gamma_{\alpha}{}^{\alpha}{}_{\omega} = 0. \tag{1.3)c}$$

The equations (1.3)bc lead to $k =$ const.

NOTE. Later, in V, we shall see that a function of g may be identified with the gravitational function [1] (which in the gravitational theory reduces to a constant). Hence, in using (1.1) to define $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, we restrict *a priori* the generality by assuming that the gravitational function reduces to a constant. This is a strong restriction from the physical point of view. Therefore, we shall not adopt the equation (1.1) as a starting point. [2]

2. The tensors $D_{\omega} g_{\lambda\mu}$ and $D_{\omega} {*g}^{\lambda\nu}$.

Instead of using (1.1) we shall adopt Einstein's condition for defining the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$,

---

[1] Gravitational functions appear in Jordan, P., (1), (2). *Cf.* also Heckmann, O. — Jordan, P. — Fricke, W., (1); Hoffmann, B., (4), (6); Havlíček, F., (1); Nariai, H., (1); Just, K., (1), (2), (3).

[2] Other possibilities for defining the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ are considered in Stephenson, A., (3). *Cf.* also Santalo, L., (2); Kilmister, C. — Stephenson, A., (2), (3); Winogradski, J., (1)–(4); Tonnelat, M., (4), (9); Bertotti, B., (1); Bandyopadhyay, G., (2); *Cf.* also Eisenhart, L. P., (2), (3), (4); Hély, J., (1); Pastori, M., (1); Singh, K., (1); Arnowitt, R., (1).

i.e.,

$$\partial_\omega g_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \Gamma_\omega{}^\alpha{}_\mu g_{\lambda\alpha} = 0. \tag{2.1)a}$$ [1] [2]

This system represents n^3 nonhomogeneous linear equations for n^3 unknowns $\Gamma_\lambda{}^\nu{}_\mu$. Hence, if this system admits a solution (which is not always the case as we shall see later), it could be obtained by an elementary algebraic device, at least for a fixed n. [3] What we are looking for in this part of the second chapter is a device which yields $\Gamma_\lambda{}^\nu{}_\mu$ in a surveyable tensorial form for any n. Finding such a device is by no means an easy task.

In subsequent considerations we shall denote by D_μ the symbol of the covariant derivative with respect to the connection defined in (2.1) and by $S_{\lambda\mu}{}^\nu$ the torsion tensor of this connection. The following Theorems will be proved simultaneously:

THEOREM (2.1). *The condition* (2.1)a *is equivalent to*

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}, \tag{2.1)b}$$

and this system can be split into

$$D_\omega h_{\lambda\mu} = 2S_{\omega(\mu}{}^\alpha g_{\lambda)\alpha} \tag{2.2)a}$$

$$D_\omega k_{\lambda\mu} = 2S_{\omega[\mu}{}^\alpha g_{\lambda]\alpha}. \tag{2.2)b}$$

THEOREM (2.2). *The tensors* $D_\omega g_{\lambda\mu}$, $D_\omega h_{\lambda\mu}$ *and* $D_\omega k_{\lambda\mu}$ *satisfy*

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[1] Compare the order of the indices ω and μ in (1.1) and (2.1)a. This order dates from a previous attempt by Einstein. *Cf.* Einstein, A., (1), (2); Hlavatý, V., (10).

[2] These equations may be derived from a variational principle. *Cf.* Schrödinger, E., (1), (2), (4); Lichnerowicz, A., (2), (3); Tonnelat, M. A., (1), (13). In Einstein, A., (6) (Appendix II), they appear as a result of the so-called "pseudo-Hermitian" symmetry (see section 4). Later, Einstein, A., (7), (8) pointed out that these equations may be derived from a variational principle. *Cf.* de Simoni, F., (2); Bose, S. N., (1); Finzi, B., (2); Winogradski, J., (3), (4), (5); Einstein, A. – Kaufman, B., (3).

[3] Such an elementary algebraic device has been used for a very special form of $g_{\lambda\mu}$ in the case of $n = 4$ by the following authors: Bandyopadhyay, G., (1); Bonnor, W. B., (1), (2), (3); Bertotti, B., (2); Clauser, E., (1); Mavrides, S., (1), (2), (4), (5); Papapetrou, A., (2); de Simoni, F., (1); Takeno, H., (1), (2); Takeno, H. – Ikeda, M. – Abe, S., (1); Vaidya, P. C., (1), (2); Lenoir, M., (1); Ghosh, N. N., (1); Ikeda, M., (2) (3); Narlikar, V. – Tiwari, R., (3); Rao, B., (1); Udeschini, P., (7).

See also the footnotes pp. 71 and 146. Almost all of these papers deal with the second class.

the following conditions:

(2.3)a $$D_\omega g_{\lambda\mu} + D_\mu g_{\lambda\omega} = 0,$$

(2.3)b $$D_{(\omega} h_{\lambda\mu)} = 0,$$

(2.3)c $$D_\lambda h_{\omega\mu} + 2D_{(\omega} k_{\mu)\lambda} = 0.$$

PROOF. Substitute into

$$D_\omega g_{\lambda\mu} = \partial_\omega g_{\lambda\mu} - \Gamma_{\lambda}{}^{\alpha}{}_{\omega} g_{\alpha\mu} - \Gamma_{\mu}{}^{\alpha}{}_{\omega} g_{\lambda\alpha}$$

for $\partial_\omega g_{\lambda\mu}$ from (2.1)a. The result is (2.1)b. The equations (2.2)a,b follow from (2.1)b and from

$$D_\omega g_{(\lambda\mu)} = D_\omega h_{\lambda\mu}; \quad D_\omega g_{[\lambda\mu]} = D_\omega k_{\lambda\mu}.$$

The condition (2.3)a follows from (2.1)b. The condition (2.3)b follows from (2.2)a. The equation (2.3)a is equivalent to

$$D_\omega h_{\lambda\mu} + D_\mu h_{\lambda\omega} - 2D_{(\omega} k_{\mu)\lambda} = 0,$$

and this equation, together with (2.3)b, leads to (2.3)c.

In the next Theorem we shall use the tensor $^*g^{\lambda\nu}$ defined by I (3.6).

THEOREM (2.3). *The equations* (2.1) *are equivalent to*

(2.4)a $$D_\omega {}^*g^{\lambda\nu} = -\, 2S_{\omega\alpha}{}^{\nu}\, {}^*g^{\lambda\alpha},$$ [1]

and this system is equivalent to

(2.4)b $$\partial_\omega {}^*g^{\lambda\nu} + \Gamma_{\alpha}{}^{\lambda}{}_{\omega} {}^*g^{\alpha\nu} + \Gamma_{\omega}{}^{\nu}{}_{\alpha} {}^*g^{\lambda\alpha} = 0.$$

PROOF. Multiply (2.1)b by $^*g^{\lambda\nu}$:

(2.5) $${}^*g^{\lambda\nu} D_\omega g_{\lambda\mu} = -\, g_{\alpha\mu} D_\omega {}^*g^{\alpha\nu} = 2S_{\omega\mu}{}^{\nu}.$$

Multiply the last equation by $^*g^{\lambda\mu}$. The result is (2.4)a. The equation (2.4)b follows easily from (2.4)a. Start with (2.4)a, and multiply this equation by $g_{\lambda\mu}$. The result is (2.5). This equation leads easily to (2.1)b.

THEOREM (2.4). *The covariant derivatives of the determinants* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are*

(2.6)a $$D_\omega \mathfrak{g} = 2S_\omega \mathfrak{g}$$

(2.6)b $$D_\omega \mathfrak{h} = 2S_\omega \mathfrak{h} - 2S_{\omega\lambda}{}^{\nu} k_{\nu}{}^{\lambda} \mathfrak{h}$$

where

(2.7) $$S_\omega \overset{\text{def}}{=} S_{\omega\lambda}{}^{\lambda}.$$

---

[1] For the solution of (2.4)a *cf.* Mavridès, S., (7).

PROOF. Put $\nu = \mu$ in (2.5). Then

$$*g^{\lambda\nu} D_\omega g_{\lambda\nu} = \mathfrak{g}^{-1} D_\omega \mathfrak{g} = 2S_\omega .$$

Multiply (2.2)a by $h^{\lambda\mu}$

$$\begin{aligned} h^{\lambda\mu} D_\omega h_{\lambda\mu} &= \mathfrak{h}^{-1} D_\omega \mathfrak{h} = 2S_{\omega\mu\lambda} h^{\mu\lambda} + 2S_{\omega(\mu}{}^{\alpha} k_{\lambda)\alpha} h^{\mu\lambda} \\ &= 2S_\omega + 2S_{\omega\lambda}{}^{\nu} k^{\lambda}{}_{\nu} . \end{aligned}$$

This equation is equivalent to (2.6)b.

3. The connection $\Gamma_\lambda{}^\nu{}_\mu$.

THEOREM (3.1). *If the system* (2.1) *admits a solution* $\Gamma_\lambda{}^\nu{}_\mu$, *this solution must be of the form*

$$\Gamma_\lambda{}^\nu{}_\mu = \{{}_\lambda{}^\nu{}_\mu\} + S_{\lambda\mu}{}^\nu + U^\nu{}_{\lambda\mu} \tag{3.1}$$

where the Christoffel symbols are defined by (7),[1] *the torsion tensor* $S_{\lambda\mu}{}^\nu$ *by* (3),[1] *and*

$$U^\nu{}_{\lambda\mu} \overset{\text{def}}{=} 2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta} . \tag{3.2}$$

Moreover, the equation (3.1) *is equivalent to* (2.2)a.

PROOF. Using (2.2)a and (2.3)b one obtains

$$\begin{aligned} \tfrac{1}{2} h^{\nu\alpha}(D_\lambda h_{\alpha\mu} + D_\mu h_{\alpha\lambda} - D_\alpha h_{\lambda\mu}) &= \\ = - h^{\nu\alpha} D_\alpha h_{\lambda\mu} &= \\ = - 2h^{\nu\alpha} S_{\alpha(\mu}{}^{\beta}(h_{\lambda)\beta} + k_{\lambda)\beta}) &= \\ = - 2S^\nu{}_{(\mu\lambda)} - U^\nu{}_{\lambda\mu} . & \end{aligned} \tag{3.3a}$$

On the other hand, the formula

$$D_\lambda h_{\alpha\mu} = \partial_\lambda h_{\alpha\mu} - \Gamma_\alpha{}^\beta{}_\lambda h_{\beta\mu} - \Gamma_\mu{}^\beta{}_\lambda h_{\alpha\beta}$$

leads to

$$\begin{aligned} \tfrac{1}{2} h^{\nu\alpha}(D_\lambda h_{\alpha\mu} + D_\mu h_{\lambda\alpha} - D_\alpha h_{\lambda\mu}) &= \\ = \{{}_\lambda{}^\nu{}_\mu\} - 2S^\nu{}_{(\mu\lambda)} - \Gamma_{(\mu}{}^\nu{}_{\lambda)} . & \end{aligned} \tag{3.3b}$$

Comparing the equations (3.3)a,b one gets

$$\Lambda_\lambda{}^\nu{}_\mu \overset{\text{def}}{=} \Gamma_{(\lambda}{}^\nu{}_{\mu)} = \{{}_\lambda{}^\nu{}_\mu\} + U^\nu{}_{\lambda\mu} . \tag{3.4}$$

---

[1] *Cf.* the introduction to this Chapter.

But

$$\Gamma_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)} + S_{\lambda\mu}{}^\nu,$$

and this equation, in conjunction with (3.4), is equivalent to (3.1). The manner in which this formula has been obtained demonstrates the equivalence of (3.1) and (2.2)a.

The equations (3.1), (3.2) reduce the investigation of $\Gamma_\lambda{}^\nu{}_\mu$ to the study of the tensor $S_{\lambda\mu}{}^\nu$. In order to know the connection it is necessary and sufficient to know the tensor $S_{\lambda\mu}{}^\nu$. Therefore, we shall look for a method yielding $S_{\lambda\mu}{}^\nu$. Our investigation is based on the tensor

$$(3.5) \qquad \begin{aligned} K_{\omega\mu\lambda} \overset{\text{def}}{=} \nabla_\omega k_{\lambda\mu} + \nabla_\mu k_{\omega\lambda} + \nabla_\lambda k_{\omega\mu} &= \\ = 3\nabla_{[\omega}k_{\lambda\mu]} + 2\nabla_\lambda k_{\omega\mu} &= \\ = 3\partial_{[\omega}k_{\lambda\mu]} + 2\nabla_\lambda k_{\omega\mu} & \end{aligned}$$

where ∇_μ is the symbolic vector of the covariant derivative with respect to $\{\lambda{}^\nu{}_\mu\}$.

THEOREM (3.2). *The tensors* $S_{\lambda\mu}{}^\nu$ *and* $U^\nu{}_{\lambda\mu}$ *are related to the tensor* $K_{\omega\mu\nu}$ *in the following way:*

$$(3.6) \qquad 2S_{\omega\mu\nu} = K_{\omega\mu\nu} - 4U_{\alpha\nu[\mu}k_{\omega]}{}^\alpha.$$

PROOF. Combining (2.1)b and (2.3)c one obtains

$$(3.7)\text{a} \qquad \begin{aligned} 2D_{(\omega}k_{\nu)\mu} + D_\mu k_{\omega\nu} = -[D_\mu h_{\nu\omega} + D_\mu k_{\nu\omega}] &= \\ = -D_\mu g_{\nu\omega} = -2S_{\mu\omega}{}^\alpha g_{\nu\alpha}. & \end{aligned}$$

On the other hand, one finds by virtue of (3.1)

$$(3.7)\text{b} \qquad \begin{aligned} D_\alpha k_{\beta\gamma} = \nabla_\alpha k_{\beta\gamma} - (S_{\beta\alpha}{}^\delta + U^\delta{}_{\beta\alpha})k_{\delta\gamma} \\ - (S_{\gamma\alpha}{}^\delta + U^\delta{}_{\gamma\alpha})k_{\beta\delta}. \end{aligned}$$

The equation (3.6) is a result of the substitution from (3.7)b into the left hand term of (3.7)a.

NOTE. Let $\overset{\circ}{D}_\mu$ be the symbol of the covariant derivative with respect to $\Lambda_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)}$. Then, by (3.4),

$$(3.8) \qquad 2\overset{\circ}{D}_\nu k_{\omega\mu} = 2\nabla_\nu k_{\omega\mu} + 4U_{\alpha\nu[\omega}k_{\mu]}{}^\alpha.$$

Substituting from (3.8) for $\nabla_\alpha k_{\beta\gamma}$ into (3.5) one obtains

$$(3.9) \qquad K_{\omega\mu\nu} = 3\partial_{[\omega}k_{\nu\mu]} + 2\overset{\circ}{D}_\nu k_{\omega\mu} + 4U_{\alpha\nu[\mu}k_{\omega]}{}^\alpha,$$

so that (3.6) yields

$$2S_{\omega\mu\nu} = 3\partial_{[\omega}k_{\nu\mu]} + 2\overset{\circ}{D}_{\nu}k_{\omega\mu}\,{}^{1}. \tag{3.10}$$

This equation is just a formal simplification of (3.6). Both equations (3.6) and (3.10) require the knowledge of $U^{\nu}{}_{\lambda\mu}$.

THEOREM (3.3). *Put*

$$X^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\text{def}}{=} \delta^{[\alpha\beta]\gamma}_{[\omega\mu]\nu} - 2\delta^{[\alpha}_{\nu}\,k_{[\omega}{}^{\beta]}k_{\mu]}{}^{\gamma} - 2\delta^{[\alpha}_{[\mu}k_{\omega]}{}^{\beta]}k_{\nu}{}^{\gamma}. \tag{3.11}$$

Then

$$2S_{\alpha\beta\gamma}X^{\alpha\beta\gamma}_{\omega\mu\nu} = K_{\omega\mu\nu}, \tag{3.12}$$

and this equation is equivalent to

$$2S_{\alpha\beta\gamma}X^{\alpha\beta\gamma}_{\omega[\mu\nu]} = \nabla_{\omega}k_{\nu\mu}. \tag{3.13}$$

PROOF. The equation (3.12) results from the substitution for U into (3.6) from (3.2). The equation (3.13) follows from (3.12) and

$$K_{\omega[\mu\nu]} = \nabla_{\omega}\,k_{\nu\mu}. \tag{3.14}$$

A simple inspection shows that (3.13) and (3.5) again lead to (3.12).

THEOREM (3.4). *The equations* (2.2)b *and* (3.12) *are equivalent.*

PROOF. The equations (3.12) and (3.6) are equivalent. From the manner in which (3.6) has been obtained (equation (3.7)a), it is evident that (3.6) and (2.2)b are equivalent.

THEOREM (3.5). *A necessary and sufficient condition for the system* (2.1) *to admit a unique solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is: The system* (3.12) *admits a unique solution $S_{\lambda\mu}{}^{\nu}$.*

PROOF. The system (2.1) is equivalent to (2.2)a,b. The system (2.2)a is equivalent to (3.1) while (2.2)b is equivalent to (3.12). The proof follows easily from these facts.

NOTE. This Theorem reduces the investigation of (2.1) to the study of (3.12), which is the main goal of subsequent sections.

4. Pseudo-Hermitian symmetry.

Put

$$\overset{+}{g}_{\lambda\mu} \overset{\text{def}}{=} g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{4.1)a}$$

$$\bar{g}_{\lambda\mu} \overset{\text{def}}{=} h_{\lambda\mu} - k_{\lambda\mu} = h_{\lambda\mu} + \bar{k}_{\lambda\mu} \tag{4.1)b}$$

---

[1] *Cf.* Stephenson, G., (3); Kursunoğlu, B., (3); Ingraham, R., (1).

where

$$\bar{k}_{\lambda\mu} \overset{\text{def}}{=} k_{\mu\lambda} = -\, k_{\lambda\mu}. \tag{4.1)c}$$

Then

$$\bar{g}_{\lambda\mu} = g_{\mu\lambda}. \tag{4.2}$$

This *formal* resemblance to the definition of Hermitian symmetry justifies the application of the name "pseudo-Hermitian symmetry" to (4.2).

The connection $\overset{+}{\Gamma}{}_{\lambda}{}^{\nu}{}_{\mu} \overset{\text{def}}{=} \Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is a function of $\overset{+}{g}_{\lambda\mu}$. Denote by $\bar{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$ *the same function* of the argument $\bar{g}_{\lambda\mu}$. Then the following Theorem holds.

THEOREM (4.1). *If the system* (3.12) *admits a unique solution, then*

$$\bar{\Gamma}_{\mu}{}^{\nu}{}_{\lambda} = \Gamma_{\lambda}{}^{\nu}{}_{\mu}, \tag{4.3}$$

so that, in particular, the term on the left hand side of (2.1)a *is pseudo-Hermitian in λ, μ.*

PROOF. Let us modify by a dash all quantities which one obtains by replacing $\overset{+}{g}_{\lambda\mu}$ by $\bar{g}_{\lambda\mu}$. Thus,

$$\begin{aligned} \bar{K}_{\omega\mu\nu} &= -\, K_{\omega\mu\nu} \\ \bar{X}^{\alpha\beta\gamma}_{\omega\mu\nu} &= X^{\alpha\beta\gamma}_{\omega\mu\nu}, \end{aligned} \tag{4.4}$$

and

$$2\bar{S}_{\alpha\beta\gamma}\bar{X}^{\alpha\beta\gamma}_{\omega\mu\nu} = \bar{K}_{\omega\mu\nu}. \tag{4.5}$$

Assume now that (3.12) admits a unique solution. Comparing (3.12) and (4.5) one obtains by virtue of (4.4)

$$\bar{S}_{\omega\mu\nu} = -\, S_{\omega\mu\nu} = S_{\mu\omega\nu}, \tag{4.6)a}$$

so that according to (3.2) and (4.1)

$$\bar{U}^{\nu}{}_{\lambda\mu} = U^{\nu}{}_{\lambda\mu}. \tag{4.6)b}$$

Hence,

$$\begin{aligned} \bar{\Gamma}_{\mu}{}^{\nu}{}_{\lambda} &= \{_{\mu}{}^{\nu}{}_{\lambda}\} + \bar{S}_{\mu\lambda}{}^{\nu} + \bar{U}^{\nu}{}_{\mu\lambda} = \\ &= \{_{\lambda}{}^{\nu}{}_{\mu}\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu} = \Gamma_{\lambda}{}^{\nu}{}_{\mu}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\partial_{\omega}\bar{g}_{\lambda\mu} - \bar{\Gamma}_{\lambda}{}^{\alpha}{}_{\omega}\bar{g}_{\alpha\mu} - \bar{\Gamma}_{\omega}{}^{\alpha}{}_{\mu}\bar{g}_{\lambda\alpha} = \\ &= \partial_{\omega}g_{\mu\lambda} - \Gamma_{\omega}{}^{\alpha}{}_{\lambda}g_{\mu\alpha} - \Gamma_{\mu}{}^{\alpha}{}_{\omega}g_{\alpha\lambda}. \end{aligned}$$

NOTE. The pseudo-Hermitian symmetry of (2.1)a (the "survivor" of the Hermitian symmetry from earlier papers by Einstein, A., (1), (2), (3) and Einstein, A. — Straus, E., (1)) was the original reason for the peculiar positions of the indices ω and μ in (2.1)a. However, Einstein A., (1) did not prove (4.3); he simply postulated it.

5. The system (3.12). Assume that a tensor $T^{\cdots}_{\omega\mu\cdots}$ is skew symmetric in $\omega\mu$, and denote by $T^{\cdots}_{\omega\mu,\cdots}$ those components for which $\omega\mu$ are in natural order ($T^{\cdots}_{\mathrm{I\,II}\cdots}$, $T^{\cdots}_{\mathrm{II\,IV}\cdots}$, etc.). Thus, $X^{\alpha\beta,\gamma}_{\omega\mu,\nu}$ are those components of the tensor $X^{\alpha\beta\gamma}_{\omega\mu\nu}$ defined by (3.11) for which $\omega\mu$, as well as $\alpha\beta$, are in natural order. Put

$$(5.1) \qquad \mathscr{X}^{\alpha\beta,\gamma}_{\omega\mu,\nu} \overset{\text{def}}{=} 2X^{\alpha\beta,\gamma}_{\omega\mu,\nu},$$

and denote by $((\mathscr{X}))$ the square matrix whose elements are $\mathscr{X}^{\alpha\beta,\gamma}_{\omega\mu,\nu}$. The rows (the columns) of $((\mathscr{X}))$ are indicated by subscripts (by superscripts). Moreover, denote by $((X))$ the square matrix of elements $X^{\alpha\beta\gamma}_{\omega\mu\nu}$, with the same significance of subscripts and superscripts as before.

LEMMA (5.1). *The matrices $((X))$ and $((\mathscr{X}))$ have the same rank.*

PROOF. Denote by $((X_1))$ the matrix which one obtains from $((X))$ by disregarding in the latter all rows and columns whose elements are zero.[1] $((X))$ and $((X_1))$ have the same rank. With each row $X^{\cdots}_{\omega\mu\nu}$ (column $X^{\alpha\beta\gamma}_{\cdots}$) of $((X_1))$ there is associated another row $X^{\cdots}_{\mu\omega\nu}$ (column $X^{\beta\alpha\gamma}_{\cdots}$). Retain only the rows $X^{\cdots}_{\omega\mu,\nu}$ (columns $X^{\alpha\beta,\gamma}_{\cdots}$) and disregard the remaining ones. The matrix $((X_2))$ so obtained has the same rank as $((X_1))$, so that $((X_2))$ and $((X))$ are of the same rank. Moreover, according to (5.1),

$$(5.2) \qquad ((\mathscr{X})) = 2((X_2)),$$

and this concludes the proof of our Lemma.

THEOREM (5.1). *If and only if the rank of $((X))$ is $m = n^2(n-1)/2$, there is a (unique) tensor*

$$(5.3) \qquad Y^{\omega\mu\nu}_{\xi\eta\zeta} = Y^{[\omega\mu]\nu}_{[\xi\eta]\zeta}$$

---

[1] In other words all rows $X^{\cdots}_{\omega\mu\nu}$ (all columns $X^{\alpha\beta\gamma}_{\cdots}$) for which $\omega = \mu$ (for which $\alpha = \beta$).

with the property

$$X^{\alpha\beta\gamma}_{\omega\mu\nu}\,Y^{\omega\mu\nu}_{\xi\eta\zeta} = \delta^{[\alpha\beta]\gamma}_{[\xi\eta]\zeta}. \tag{5.4}$$

The tensor Y *is defined as follows: Put*

$$\mathscr{Z}^{\omega\mu,\nu}_{\xi\eta,\zeta} \overset{\text{def}}{=} \frac{\partial}{\partial \mathscr{X}^{\xi\eta,\zeta}_{\omega\mu,\nu}} \ln \operatorname{Det}\,((\mathscr{X})), \tag{5.5)a}$$

$$Z^{\omega\mu,\nu}_{\xi\eta,\zeta} \overset{\text{def}}{=} \tfrac{1}{2}\mathscr{Z}^{\omega\mu,\nu}_{\xi\eta,\zeta}, \tag{5.5)b}$$

$$Y^{[\omega\mu]\nu}_{[\xi\eta]\zeta} \overset{\text{def}}{=} Z^{\omega\mu,\nu}_{\xi\eta,\zeta}.\,{}^{1} \tag{5.5)c}$$

Then $Y^{\omega\mu\nu}_{\xi\eta\zeta}$ *is defined by* (5.3).[2]

PROOF. Assume that the rank of $((X))$ is m. Then $((\mathscr{X}))$ is of the same rank, and we have according to (5.5)a

$$\mathscr{X}^{\alpha\beta,\gamma}_{\omega\mu,\nu}\mathscr{Z}^{\omega\mu,\nu}_{\xi\eta,\zeta} = \delta^{\alpha\beta,\gamma}_{\xi\eta,\zeta}; \tag{5.6}$$

or, by virtue of (5.1), (5.3) and (5.5)b,c,

$$4X^{\alpha\beta,\gamma}_{\omega\mu,\nu}\,Y^{\omega\mu,\nu}_{\xi\eta,\zeta} = 2\,X^{\alpha\beta,\gamma}_{\omega\mu\nu}\,Y^{\omega\mu\nu}_{\xi\eta,\zeta} = \delta^{\alpha\beta,\gamma}_{\xi\eta,\zeta}. \tag{5.7}$$

On the other hand,

$$\begin{aligned} X^{\alpha\beta,\gamma}_{\omega\mu\nu} &\overset{\text{def}}{=} X^{\alpha\beta\gamma}_{\omega\mu\nu} \\ Y^{\omega\mu\nu}_{\xi\eta,\zeta} &\overset{\text{def}}{=} Y^{\omega\mu\nu}_{\xi\eta\zeta} \\ \delta^{\alpha\beta,\gamma}_{\xi\eta,\zeta} &\overset{\text{def}}{=} 2\delta^{[\alpha\beta]\gamma}_{[\xi\eta]\zeta}. \end{aligned} \tag{5.8}$$

Substituting from (5.8) into (5.7) one obtains (5.4).[3] The remaining part of the proof is obvious.

THEOREM (5.2). *A necessary and sufficient condition for the system* (2.1) *to admit a unique solution is: The matrix* $((X))$ *is of rank* $m = n^2(n-1)/2$. *If this condition is satisfied, then*

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{\lambda{}^{\nu}{}_{\mu}\} + (\tfrac{1}{2}K_{\alpha\beta\gamma}\,Y^{\alpha\beta\gamma}_{\lambda\mu\omega} + K_{\alpha\beta\gamma}\,Y^{\alpha\beta\gamma}_{\omega(\lambda|\delta|}\,k_{\mu)}{}^{\delta})h^{\omega\nu}. \tag{5.9}$$

---

[1] This equation defines the components $Y^{[\omega\mu]\nu}_{[\xi\eta]\zeta}$ for which $\omega\mu$ as well as $\xi\eta$ are in natural order. The remaining components may be obtained from

$$Y^{[\omega\mu]\nu}_{[\xi\eta]\zeta} = -\,Y^{[\mu\omega]\nu}_{[\xi\eta]\zeta} = Y^{[\mu\omega]\nu}_{[\eta\xi]\zeta} = -\,Y^{[\omega\mu]\nu}_{[\eta\xi]\zeta}.$$

[2] *Cf.* Theorems (12.3) and (16.1) for $n = 4$.

[3] Strictly speaking one obtains (5.4) where $\alpha\beta$ as well as $\xi\eta$ are in natural order. Due to (5.3) and to a similar equation for X, (5.4) is then valid for all pairs of indices $\alpha\beta$ and $\xi\eta$.

PROOF. A necessary and sufficient condition for (3.12) to admit a unique solution $S_{\omega\mu\nu}$ is that the matrix $((X))$ be of rank $m = n^2(n-1)/2$: If and only if this condition is satisfied, we obtain from (3.12) and (5.4)

$$S_{\omega\mu\nu} = \tfrac{1}{2}K_{\alpha\beta\gamma}Y^{\alpha\beta\gamma}_{\omega\mu\nu}. \tag{5.10}$$

The remaining part of the proof follows from Theorems (3.1) and (3.5).

NOTE. The solution (5.9) holds for any n. [1] It is not too difficult in the case of $n = 4$ to express Y in terms of $g_{\lambda\mu}$. However, even in this simple case the solution results in a much too complicated form. [2] Therefore, we shall find in the next part of this chapter another way of expressing it. The method will be outlined only for $n = 4$ and will also comprise the degenerate cases excluded by Theorem (5.2) (see the next chapter).

THEOREM (5.3). *The requirement for the existence and uniqueness of the solution* $\Gamma_\lambda{}^\nu{}_\mu$ *of* (2.1) *does not impose any restriction on* $g_{\lambda\mu}$ *in the form of an equation.*

The proof follows at once from the previous Theorem.

NOTE. *Cf.* Theorem (1.1).

THEOREM (5.4). *A sufficient condition that the system* (2.1) *admit either more than one solution* $\Gamma_\lambda{}^\nu{}_\mu$ *or none is*

$$\mathfrak{g} = 0. \tag{5.11}$$

PROOF. Assume that (5.11) is satisfied. Then there are at least two vectors A^λ, B^λ, such that

$$A^\lambda g_{\lambda\mu} = 0 \tag{5.12)a}$$

$$g_{\mu\lambda}B^\lambda = 0. \tag{5.12)b}$$

Let $C^\nu \neq 0$ be an arbitrary vector and put

$$A^{\omega\lambda\mu} \overset{\text{def}}{=} C^\omega A^\lambda B^\mu.$$

Then we have not only

$$A^{\omega\lambda\mu}g_{\lambda\mu} = 0, \tag{5.13)a}$$

---

[1] Another method of solution for general n is in Tonnelat, M. A., (2), (5), (11). *Cf.* also Schouten, J. A., (1). However, this latter method is incomplete.

[2] *Cf.* Tonnelat, M. A., (10), (12), (13); Straus, E., (1).

but also

$$S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha} A^{\omega\lambda\mu} = 0. \tag{5.13)b}$$

The equation (5.13)a yields by virtue of (5.12)

$$A^{\omega\lambda\mu} D_{\omega} g_{\lambda\mu} = - g_{\lambda\mu} D_{\omega} A^{\omega\lambda\mu} = 0. \tag{5.13)c}$$

Hence, we have, *independently of* (2.1)b,

$$A^{\omega\lambda\mu}(D_{\omega} g_{\lambda\mu} - 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}) = 0. \tag{5.14}$$

Therefore, the equations (2.1)b are not independent. If they are consistent, then they will admit more than one solution, $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. If they are not consistent, there is no solution.

NOTE. The condition (5.11) is not a necessary one as we shall see later (*Cf.* Section (14)).

THEOREM (5.5). *A necessary condition for the system* (2.1) *to admit exactly one solution is*

$$\mathfrak{g} \neq 0. \tag{5.15}$$

PROOF. Assume that (2.1) admits exactly one solution. Then according to the previous Theorem we cannot have (5.11). Hence, we must have (5.15).

NOTE I. The condition (5.15) is not a sufficient one as we shall see later (*Cf.* Section (14)).

NOTE II. Although the equation (5.9) solves our problem of finding $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, there are many cases where the tensor Y is not needed. Some of these cases will be presented in the next two sections.

6. The tensor $S_{\lambda\mu}{}^{\nu}$.

In the next Theorem we shall assume that the tensor

$$X_{\mu}{}^{\nu} \overset{\text{def}}{=} \delta_{\mu}^{\nu}(n - 1 + 4K) + {}^{(2)}k_{\mu}{}^{\nu} \tag{6.1)a}$$

is of rank n, so that there is a unique tensor $Y_{\mu}{}^{\nu}$ satisfying the condition

$$Y_{\mu}{}^{\alpha} X_{\alpha}{}^{\nu} = \delta_{\mu}^{\nu}. \tag{6.1)b}$$

Moreover, we shall use the abbreviation

$$K_{\mu}{}^{\nu} \overset{\text{def}}{=} \delta_{\mu}^{\nu} - {}^{(2)}k_{\mu}{}^{\nu}. \tag{6.2}$$

THEOREM (6.1). *A necessary and sufficient condition for the system* (3.12) *to admit exactly one solution* $S_{\omega\mu\nu}$ *of the form*

$$S_{\omega\mu\nu} = \frac{2}{n-1} S_{[\omega} h_{\mu]\nu} \tag{6.3}$$

is

$$K_{\omega\mu}{}^{\nu} = 4 Y_{[\omega}{}^{\gamma} K_{\mu]}{}^{\nu} \nabla_{\alpha} k_{\gamma}{}^{\alpha}. \tag{6.4}$$

If this condition is satisfied, then

$$S_{\lambda} = (n-1) Y_{\lambda}{}^{\alpha} \nabla_{\beta} k_{\alpha}{}^{\beta}. \tag{6.5}$$

PROOF. Substituting from (6.3) into (3.12) one obtains

$$4 S_{[\omega} K_{\mu]}{}^{\nu} = (n-1) K_{\omega\mu}{}^{\nu}, \tag{6.6a}$$

and this equation yields

$$-\frac{1}{n-1} S_{\alpha} X_{\mu}{}^{\alpha} = \nabla_{\alpha} k_{\mu}{}^{\alpha} \tag{6.6)b}$$

equivalent to (6.5). The equation (6.4) results from the substitution of (6.5) into (6.6)a. Assume now that (6.4) is satisfied. Define the tensor $S_{\omega\mu\nu}$ by (6.3) and (6.5), and substitute it into (3.12). This substitution reduces the latter equation to (6.4) which is satisfied by our assumption. Assume now that (3.12) admits another solution

$$^{*}S_{\omega\mu\nu} = \frac{2}{n-1} {}^{*}S_{[\omega} h_{\mu]\nu} \tag{6.7)a}$$

$$^{*}S_{\omega} \neq S_{\omega}. \tag{6.7)b}$$

Applying the previous method to this solution one obtains

$$^{*}S_{\omega} = (n-1) Y_{\omega}{}^{\alpha} \nabla_{\beta} k_{\alpha}{}^{\beta} = S_{\omega},$$

and this result contradicts (6.7)b.

COROLLARY (6.1). *In the case mentioned in the previous Theorem one has*

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{\lambda{}^{\nu}{}_{\mu}\} + (Y_{\lambda}{}^{\alpha} g_{\beta\mu} - Y_{\mu}{}^{\alpha} g_{\lambda\beta}) h^{\beta\nu} \nabla_{\gamma} k_{\alpha}{}^{\gamma}. \tag{6.8}$$

PROOF. This equation results from (6.5), (6.3), (3.2) and (3.1).

COROLLARY (6.2). *In the case* (6.8), *the requirements*

$$\text{a) } S_{\mu} = 0 \qquad \text{b) } \nabla_{\alpha} k_{\mu}{}^{\alpha} = 0 \tag{6.9}$$

are equivalent and reduce (6.8) *to*

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{\lambda{}^{\nu}{}_{\mu}\}. \tag{6.10}$$

PROOF. The equivalence of (6.9)a and (6.9)b follows from (6.5). The remaining part of the proof is obvious.[1]

NOTE. Later we shall see that one of the field equations is

$$S_{\lambda} = 0. \tag{6.11}$$

Therefore, the previous Corollary is very important for the study of this field equation.

The last two Theorems of this section will be very important for the study of the electromagnetic field.

THEOREM (6.2). *We have*

$$2S_{[\omega\mu\lambda]} = -\,\partial_{[\omega}k_{\mu\lambda]}. \tag{6.12}$$

PROOF. First one obtains from (2.2)b

$$\begin{aligned} D_{[\omega}k_{\lambda\mu]} &= 2S_{[\omega\mu}{}^{\alpha}g_{\lambda]\alpha} = \\ &= 2S_{[\omega\mu\lambda]} + 2S_{[\omega\mu}{}^{\alpha}k_{\lambda]\alpha}. \end{aligned} \tag{6.13)a}$$

On the other hand,

$$\begin{aligned} D_{[\omega}k_{\lambda\mu]} &= \partial_{[\omega}k_{\lambda\mu]} - k_{\alpha[\mu}\Gamma_{\lambda}{}^{\alpha}{}_{\omega]} - \Gamma_{[\mu}{}^{\alpha}{}_{\omega}k_{\lambda]\alpha} \\ &= \partial_{[\omega}k_{\lambda\mu]} + 2S_{[\omega\mu}{}^{\alpha}k_{\lambda]\alpha}. \end{aligned} \tag{6.13)b}$$

The equation (6.12) results from (6.13)a and (6.13)b.

THEOREM (6.3). *The tensor* $*k^{\lambda\nu}$[2] *satisfies the condition*

$$\partial_{\lambda}(\sqrt{|\mathfrak{g}|}\,{*k^{\lambda\nu}}) = -\,S_{\lambda}\sqrt{|\mathfrak{g}|}\,{*h^{\lambda\nu}}. \tag{6.14}$$

PROOF. A simple inspection based on I(3.7)a, (2.6)a and (2.4)a shows

$$D_{\lambda}(\sqrt{|\mathfrak{g}|}\,{*k^{\lambda\nu}}) = \sqrt{|\mathfrak{g}|}(-\,S_{\alpha}{*h^{\alpha\nu}} + 2S_{\alpha}{*k^{\alpha\nu}} - S_{\lambda\alpha}{}^{\nu}{*k^{\lambda\alpha}}). \tag{6.15}$$

On the other hand,

$$\begin{aligned} D_{\lambda}(\sqrt{|\mathfrak{g}|}\,{*k^{\lambda\nu}}) &= \partial_{\lambda}(\sqrt{|\mathfrak{g}|}\,{*k^{\lambda\nu}}) + \sqrt{|\mathfrak{g}|}\{\Gamma_{\alpha}{}^{\lambda}{}_{\lambda}{*k^{\alpha\nu}} + \Gamma_{\alpha}{}^{\nu}{}_{\lambda}{*k^{\lambda\alpha}} - \Gamma_{\alpha}{}^{\alpha}{}_{\lambda}{*k^{\lambda\nu}}\} \\ &= \partial_{\lambda}(\sqrt{|\mathfrak{g}|}\,{*k^{\lambda\nu}}) + \sqrt{|\mathfrak{g}|}(2S_{\alpha}{*k^{\alpha\nu}} + S_{\alpha\lambda}{}^{\nu}{*k^{\lambda\alpha}}). \end{aligned}$$

---

[1] *Cf.* (2.7). Another example where Y is not needed is in Hlavatý, V., (7).

[2] As in the first chapter, $*k^{\lambda\nu}$ is the skew symmetric part of the tensor $*g^{\lambda\nu}$ satisfying the condition I (3.6).

Comparing the last two equations we obtain (6.14).

NOTE I. If we denote by $\overset{\circ}{D}_\lambda$ the symbol of the covariant derivative with respect to $\Lambda_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)}$, then obviously

$$\overset{\circ}{D}_\lambda \sqrt{|\mathfrak{g}|}\, {}^*k^{\lambda\nu} = \partial_\lambda \sqrt{|\mathfrak{g}|}\, {}^*k^{\lambda\nu}$$

or, according to (6.14),

$$\overset{\circ}{D}_\lambda \sqrt{|\mathfrak{g}|}\, {}^*k^{\lambda\nu} = - S_\lambda \sqrt{|\mathfrak{g}|}\, {}^*h^{\lambda\nu}.$$

If, in particular, $S_\lambda = 0$, we have

$$\overset{\circ}{D}_\lambda \sqrt{|\mathfrak{g}|}\, {}^*k^{\lambda\nu} = 0.$$ [1]

NOTE II. This Theorem will enable us later to identify the electromagnetic field.

7. The tensor $U^\nu{}_{\lambda\mu}$. The tensor $U^\nu{}_{\lambda\mu}$ is defined by (3.2). On the other hand, it can be computed directly and independently of $S_{\lambda\mu}{}^\nu$.

THEOREM (7.1). *The tensor $U^\nu{}_{\lambda\mu}$ satisfies the following conditions:*

(7.1)a $$U_{\nu[\lambda\mu]} = 0,$$

(7.1)b $$U_{(\nu\lambda\mu)} = 0,$$

(7.1)c $$2U_\mu \overset{\text{def}}{=} 2U^\alpha{}_{\alpha\mu} = \partial_\mu \ln g$$

and

(7.2) $$U_{\alpha\beta\gamma}\{\delta^{\alpha\beta\gamma}_{\nu(\lambda\mu)} - 2k_{(\mu}{}^\beta k_{\lambda)}{}^\alpha \delta^\gamma_\nu + 2k_\nu{}^\alpha k_{(\mu}{}^\beta \delta^\gamma_{\lambda)}\} = - k_{(\mu}{}^\beta K_{\lambda)\nu\beta}.$$

PROOF. The equations (7.1)a,b follow directly from (3.2). The equation (7.2) results from the substitution for $S_{\omega\mu\nu}$ from (3.6) into (3.2). In order to prove (7.1)c, start with (2.6) a and (3.1)

$$2S_\omega \mathfrak{g} = D_\omega \mathfrak{g} = \mathfrak{g}(\partial_\omega \ln \mathfrak{g} - 2\Gamma_\alpha{}^\alpha{}_\omega) =$$
$$= \mathfrak{g}\{\partial_\omega \ln \mathfrak{g} - \partial_\omega \ln \mathfrak{h} + 2S_\omega - 2U_\omega\}.$$

This equation leads at once to (7.1)c.

The equations (7.2) may be looked upon as a set of $n^2(n+1)/2$ linear non-homogeneous equations for $n^2(n+1)/2$ unknowns $U_{\nu(\lambda\mu)}$. Denote by $\mathfrak{d}$ its determinant.

---

[1] *Cf.* Pastori, M., (2).

THEOREM (7.2). *Under present conditions*[1] *the determinant* $\mathfrak{d}$ *does not vanish identically. If the equations of the system* (7.2) *are taken in an appropriate order, then*

$$\mathfrak{d} = 1 + V \tag{7.3}$$

where V *is a polynomial in* $k_\lambda{}^\nu$ *of degree* ≥ 2.

PROOF. If $k_{\lambda\mu} = 0$, then with the appropriate order of the equations (7.2) we have $\mathfrak{d} = 1$. Hence, for reasons of continuity we must have (7.3) for sufficiently small $k_{\lambda\mu} \neq 0$, where V is a polynomial in $k_\lambda{}^\nu$. Moreover, we must have $V = 0$ for $k_{\lambda\mu} = 0$. Hence, V does not contain any constant term, and the lowest degree of its terms is obviously $d \geq 2$. On the other hand, the only allowable restriction is either $k = 0$ or $k = 0$, $K = 0$. Hence, under these present conditions[1] $\mathfrak{d}$ as defined by (7.3) does not vanish identically.

COROLLARY (7.1).[2] *The equation*

$$k_{(\mu}{}^\beta K_{\lambda)\nu\beta} = 0 \tag{7.4)a}$$

is necessary and sufficient in order that under present conditions[1]

$$U_{\nu\lambda\mu} = 0. \tag{7.4)b}$$

NOTE. If (7.4)b holds, then we have according to (3.6)

$$S_{\omega\mu\nu} = \tfrac{1}{2} K_{\omega\mu\nu}. \tag{7.5}$$

COROLLARY (7.2). *Whenever* (7.4) *holds, then* g *and* K *are constant.*

PROOF. First we have according to I (1.5)c

$$4\partial_\omega K = \partial_\omega k_{\alpha\beta} k^{\alpha\beta} = -\,2k^{\beta\alpha} \nabla_\omega\, k_{\alpha\beta}. \tag{7.6)a}$$

On the other hand, by virtue of (7.4)a, (7.6)a and (3.5)

$$0 = k^{\mu\beta} K_{\mu\nu\beta} = k^{\mu\beta} \nabla_\nu\, k_{\mu\beta} = 2\partial_\nu K. \tag{7.6)b}$$

---

[1] In general (*i.e.*, without further restrictions) $k_{\lambda\mu}$ belongs to the first class. So far the only allowable restrictions imposed on $k_{\lambda\mu}$ reduce to the requirement that $k_{\lambda\mu}$ belong to one of the last two classes. This situation is described in the text by the words "present conditions."

[2] We exclude here cases where $V = -1$ in some particular points.

Therefore, according to (7.6)a,b K is constant. The constancy of g follows from (7.4)b and (7.1)c.

There is an interesting relationship between $h_{\lambda\mu}$ and $U_{\nu\lambda\mu}$. In order to establish it we denote as usual by $\overset{\circ}{D}_\omega$ the symbol of the covariant derivative with respect to the connection $\Lambda_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)}$.

THEOREM (7.3). *We have*

$$\overset{\circ}{D}_\omega h_{\lambda\mu} = U_{\omega\lambda\mu}. \tag{7.7}$$

PROOF. Start with the expression for $\overset{\circ}{D}_\omega h_{\lambda\mu}$

$$\begin{aligned}\overset{\circ}{D}_\omega h_{\lambda\mu} &= \nabla_\omega h_{\lambda\mu} - U^\alpha{}_{\lambda\omega} h_{\alpha\mu} - U^\alpha{}_{\mu\omega} h_{\lambda\alpha} = \\ &= -(U_{\mu\lambda\omega} + U_{\lambda\mu\omega}) = -(U_{\mu\lambda\omega} + U_{\lambda\omega\mu}).\end{aligned} \tag{7.8}$$

The equation (7.7) follows from (7.8) and (7.1)b.

The tensor $U_{\nu\lambda\mu}$ is connected with the theory of paths. *A path of* $\Gamma_\lambda{}^\nu{}_\mu$ is a curve $x^\nu(p)$ whose tangential vector dx^ν/dp satisfies the relation

$$\frac{dx^{[\lambda}}{dp}\left(\frac{d^2x^{\nu]}}{dp^2} + \Gamma_\alpha{}^{\nu]}{}_\beta \frac{dx^\alpha}{dp}\frac{dx^\beta}{dp}\right) = 0. \tag{7.9}$$

A path of $\{{}_\lambda{}^\nu{}_\mu\}$ is defined in a similar way and is called *a geodesic.*

THEOREM (7.4). *A necessary and sufficient condition that every path of* $\Gamma_\lambda{}^\nu{}_\mu$ *be a geodesic* (*and vice versa*) *is* (7.4)a.

PROOF. The equation (7.9) is equivalent to

$$\frac{dx^{[\lambda}}{dp}\left[\frac{d^2x^{\nu]}}{dp^2} + \{{}_\alpha{}^{\nu]}{}_\beta\}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp} + U^{\nu]}{}_{\alpha\beta}\frac{dx^\alpha}{dp}\frac{dx^\beta}{dp}\right] = 0. \tag{7.10}$$

Hence, a necessary and sufficient condition for our requirement to be satisfied is

$$U^\nu{}_{\lambda\mu} v^\lambda v^\mu = \rho v^\nu, \tag{7.11}$$

where ρ is a suitably chosen scalar and v^ν an arbitrary vector. This equation yields in turn

$$\frac{\partial}{\partial v^\lambda} U^\nu{}_{\alpha\beta} v^\alpha v^\beta = 2U^\nu{}_{\lambda\mu} v^\mu = \frac{\partial \rho}{\partial v^\lambda} v^\nu + \rho\delta^\nu_\lambda \tag{7.12}$$

$$\frac{\partial^2}{\partial v^\lambda \partial v^\mu} U^\nu{}_{\alpha\beta} v^\alpha v^\beta = 2U^\nu{}_{\lambda\mu} = \frac{\partial^2 \rho}{\partial v^\lambda \partial v^\mu} v^\nu + \frac{\partial \rho}{\partial v^\lambda}\delta^\nu_\mu + \frac{\partial \rho}{\partial v^\mu}\delta^\nu_\lambda.$$

The middle member in the last equation does not contain v^ν. Hence, we must have

$$\frac{\partial^2 p}{\partial v^\lambda \partial v^\mu} = 0,$$

so that

$$\frac{\partial p}{\partial v^\lambda} = 2p_\lambda$$

where p_λ is independent of v^ν. Then the last equation of (7.12) reduces to

$$U^\nu{}_{\lambda\mu} = p_\lambda \delta^\nu_\mu + p_\mu \delta^\nu_\lambda. \tag{7.13}$$

Applying (7.1)b to (7.13), we obtain $p_\mu = 0$ and this implies (7.4)b. The remaining part of the proof follows from the Corollary (7.1).

A path (a geodesic) is termed *minimal* if its tangential vector satisfies the condition

$$h_{\lambda\mu} \frac{dx^\lambda}{dp} \frac{dx^\mu}{dp} = 0. \tag{7.14}$$

THEOREM (7.5). *Consider the following set of conditions:*

I) *The tensor* $U^\nu{}_{\lambda\mu}$ *is defined by*

$$-(n-1)U^\nu{}_{\lambda\mu} = 2U^\nu h_{\lambda\mu} - U_\mu \delta^\nu_\lambda - U_\lambda \delta^\nu_\mu, \tag{7.15)a}$$

$$U_\mu \neq 0. \tag{7.15)b}$$

II) *If*

$$U^\alpha (D_\alpha U^{[\nu}) U^{\lambda]} = 0, \tag{7.16)a}$$

then

$$U^\mu U^\lambda h_{\lambda\mu} = 0. \tag{7.16)b}$$

The conditions (7.15) *and* (7.16) *are necessary and sufficient in order that the following requirements be satisfied:*

a) *Every minimal path of* $\Gamma_\lambda{}^\nu{}_\mu$ *is at the same time a minimal geodesic and vice versa.*

b) *Every other path is different from a geodesic and vice versa.*

PROOF. Consider first a tensor $Y_{\lambda\mu}{}^\nu = Y_{(\lambda\mu)}{}^\nu$ such that for every vector v^ν we have

$$Y_{\lambda\mu}{}^\nu v^\lambda v^\mu = p v^\nu. \tag{7.17)a}$$

Using the method which led to (7.13), one gets

$$Y_{\lambda\mu}{}^{\nu} = Y_{\mu}\delta^{\nu}_{\lambda} + Y_{\lambda}\delta^{\nu}_{\mu} \tag*{(7.17)b}$$

with an arbitrary vector Y_{μ}. Consider now the possibility of there being a tensor $X_{\lambda\mu}{}^{\nu} = X_{(\lambda\mu)}{}^{\nu}$ such that for any null vector w^{ν}, and only for null vectors, we have

$$X_{\lambda\mu}{}^{\nu}w^{\lambda}w^{\mu} = 0. \tag*{(7.18)a}$$

This requirement is satisfied if and only if for any vector u_{ν} the tensor $u_{\nu}X_{\lambda\mu}{}^{\nu}$ is proportional to $h_{\lambda\mu}$; in other words, if

$$X_{\lambda\mu}{}^{\nu} = 2h_{\lambda\mu}X^{\nu} \tag*{(7.18)b}$$

where X^{ν} is an arbitrary vector. Hence, in order that requirement a) be satisfied, it is necessary and sufficient that

$$U^{\nu}{}_{\lambda\mu} = 2h_{\lambda\mu}X^{\nu} + \delta^{\nu}_{\lambda}Y_{\mu} + \delta^{\nu}_{\mu}Y_{\lambda}. \tag*{(7.19)a}$$

Applying (7.1)b to this equation we obtain

$$X_{\nu} = -\,Y_{\nu} \tag*{(7.19)b}$$

so that (7.19)a reduces to

$$U^{\nu}{}_{\lambda\mu} = 2h_{\lambda\mu}X^{\nu} - \delta^{\nu}_{\lambda}X_{\mu} - \delta^{\nu}_{\mu}X_{\lambda}, \tag*{(7.19)c}$$

and this equation yields

$$U_{\mu} = -\,(n-1)X_{\mu}$$

so that (7.19)c is equivalent to (7.15)a. In order that requirement b) be satisfied, it is necessary that (7.15)b hold. If (7.15)a,b are satisfied, then requirement b) holds for all curves with the possible exception of curves C, whose tangential vector is U^{ν}. For such a curve we have from (7.15)a

$$U^{\nu}{}_{\lambda\mu}U^{\lambda}U^{\mu} = 0. \tag*{(7.20)}$$

Hence, if C is a path (*i.e.*, if (7.16)a holds), then it is also a geodesic. Therefore, for requirement b) to be satisfied without exception, we have to assume (7.16)b also. Then C is a minimal path which is at the same time a minimal geodesic as required by a).

NOTE I. The special form of $U^{\nu}{}_{\lambda\mu}$ given by (7.15)a has a physical significance which has been considered in detail in Hlavatý, V., (17), (18). *Cf.* also Clauser, E., (2).

NOTE II. If (7.15) holds, then (3.6) reduces to

$$S_{\omega\mu\nu} = \tfrac{1}{2}K_{\omega\mu\nu} + \frac{1}{n-1}[4U_\alpha h_{\nu[\mu} k_{\omega]}{}^\alpha - 2U_\nu k_{\omega\mu} - 2U_{[\mu} k_{\omega]\nu}]. \tag{7.21}$$

In the next Theorem we shall again use the abbreviation (6.2).

THEOREM (7.6)a. *A necessary and sufficient condition for* (7.15) *to be satisfied by the tensor* $U^\nu{}_{\lambda\mu}$ *of* (3.2) *is*

$$(n-1)\, k_{(\mu}{}^\beta K_{\lambda)\nu\beta} = 2[\delta^\rho_\nu K_{\lambda\mu} - \delta^\rho_{(\lambda} K_{\mu)\nu}]U_\rho \neq 0. \tag{7.22)a}$$

PROOF. Assume that (7.15)a holds, and substitute from it for $U_{\nu\lambda\mu}$ into (7.2). The result of this substitution is the equation (7.22)a. Assume that (7.15)b holds, so that $U_{\nu\lambda\mu} \neq 0$. Then the left hand term in (7.22)a must not vanish (*Cf.* Corollary (7.1)) so that the inequality (7.22)a must hold. Assume now that (7.22)a holds, and define a tensor $U_{\nu\lambda\mu}$ by (7.15). Substituting this tensor into (7.2) one obtains (7.22)a which is satisfied according to our assumption. If we define $S_{\omega\mu\nu}$ by (7.21), then a simple inspection shows that the tensor $U_{\nu\lambda\mu}$ defined by (7.15) satisfies (3.2) [1].

THEOREM (7.6)b. *If the conditions* (7.22)a *are satisfied, then the corresponding connection* $\Gamma_\lambda{}^\nu{}_\mu$ *is given by*

$$\Gamma_\lambda{}^\nu{}_\mu = \{\lambda{}^\nu{}_\mu\} + \tfrac{1}{2}K_{\lambda\mu}{}^\nu + \frac{2}{n-1}U_\alpha\{2\delta^\nu_{[\mu} k_{\lambda]}{}^\alpha - \delta^\alpha_{[\mu} k_{\lambda]}{}^\nu - h^{\alpha\nu} g_{\lambda\mu} + \delta^\nu_{(\lambda}\delta^\alpha_{\mu)}\}. \tag{7.22)b}$$

PROOF. If (7.22)a is satisfied, then (7.15) and (7.21) hold. Substituting for $U^\nu{}_{\lambda\mu}$ and $S_{\lambda\mu}{}^\nu$ from these equations into (3.1), one obtains (7.22)b.

We conclude this section by a Theorem which describes the relationship between the vector S_λ defined by (2.7) and $U_{\nu\lambda\mu}$.

THEOREM (7.7). *We have*

$$S_\omega = \nabla_\alpha k_\omega{}^\alpha + 2U_\alpha k_\omega{}^\alpha + U_{\alpha\beta\omega} k^{\beta\alpha}. \tag{7.23}$$

PROOF. The equation (3.6) yields

$$2S_{\omega\mu\nu}h^{\mu\nu} = 2S_\omega = K_{\omega\alpha}{}^\alpha - 2U_{\alpha\nu\mu} k_\omega{}^\alpha h^{\nu\mu} + 2U_{\alpha\beta\omega} k^{\beta\alpha}.$$

---

[1] The equation (10.8)b of the corresponding Theorem (10.3) in Hlavatý, V., (7) contains an obvious misprint.

This equation together with (7.1)b,c leads to (7.23).

NOTE. This Theorem enables us to find some geometrical consequences of the field equation (6.11), which we shall discuss in the fourth chapter. Assume that $n = 4$ and $k_{\mu\lambda}$ is of the third class. Then by I (3.3)b $g = 1$, and consequently, by (7.1)c

$$U_{\mu} = 0. \tag{7.24}$$

Furthermore, if (6.11) holds, then we have according to (6.14)

$$\partial_{\lambda}\sqrt{|\mathfrak{g}|}\,{}^{*}k^{\lambda\nu} = 0,$$

and this equation reduces by virtue of I (3.9)b, I (1.5)b and I (1.6)c to

$$0 = \partial_{\lambda}\sqrt{|\mathfrak{h}|}\,k^{\lambda\nu} = \nabla_{\lambda}\sqrt{|\mathfrak{h}|}\,k^{\lambda\nu} = \sqrt{|\mathfrak{h}|}\,\nabla_{\lambda}\,k^{\lambda\nu}.$$

Hence, the equation (7.23) reduces in our case to

$$U_{\alpha\beta\omega}k^{\beta\alpha} = 0. \tag{7.25}$$

8. Approximations. In the previous two sections we presented some examples where the tensor $S_{\omega\mu\nu}$ (and therefore, also the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$) could be easily computed. In this section we shall exhibit a method which leads quickly to an approximation of $S_{\omega\mu\nu}$ with an *a priori* given degree of accuracy. The assumption needed for this method is that the field $k_{\lambda\mu}$ is so "weak" that the products of $k_{\lambda\mu}$ and $\partial_{\omega}k_{\lambda\mu}$ of degree p (≥ 2) may be neglected. In connection with this assumption we shall use the equality sign $\overset{p}{=}$ whenever we shall need to stress that an equation holds if the above-mentioned products of degree $\geq p$ are neglected. Moreover, we shall make use of the following tensors:

$$\overset{(1)}{V}{}^{\alpha\beta\gamma}_{\nu\lambda\mu} \overset{\text{def}}{=} 2k_{(\mu}{}^{(\beta}k_{\lambda)}{}^{|\alpha|}\delta^{\gamma)}_{\nu} - 2k_{\nu}{}^{\alpha}k_{(\mu}{}^{(\beta}k_{\lambda)}{}^{\gamma)} \tag{8.1)a}$$

$$\overset{(s)}{V}{}^{\alpha\beta\gamma}_{\nu\lambda\mu} \overset{\text{def}}{=} \overset{(s-1)}{V}{}^{\xi\eta\zeta}_{\nu\lambda\mu}\,\overset{(1)}{V}{}^{\alpha\beta\gamma}_{\xi\eta\zeta}, \quad s = 2, 3, \ldots \tag{8.1)b}$$

$$Z^{\alpha\beta\gamma}_{\nu\lambda\mu} \overset{\text{def}}{=} \delta^{\alpha\beta\gamma}_{\nu(\lambda\mu)} - \overset{(1)}{V}{}^{\alpha\beta\gamma}_{\nu\lambda\mu} \tag{8.1)c}$$

$$H_{\nu\lambda\mu} \overset{\text{def}}{=} K_{\nu(\lambda}{}^{\alpha}k_{\mu)\alpha} \tag{8.1)d}$$

so that (7.2) may be written

$$U_{\alpha\beta\gamma}Z^{\alpha\beta\gamma}_{\nu\lambda\mu} = H_{\nu\lambda\mu}. \tag{8.2}$$

THEOREM (8.1). *Under present conditions*[1] *we have*

$$U_{\nu\lambda\mu} \overset{2}{=} 0. \tag{8.3}$$

PROOF. Under present conditions[1] the determinant $\mathfrak{d}$ does not vanish. Therefore, (8.2) is equivalent to

$$U_{\nu\lambda\mu} = H_{\alpha\beta\gamma} P^{\alpha\beta\gamma}_{\nu\lambda\mu}/\mathfrak{d} \tag{8.4}$$

where P is a polynomial in $k_{\lambda\mu}$, and the lowest degree in $k_{\lambda}{}^{\nu}$ of its terms is $d \geq 2$. On other hand, we see from (7.3) that

$$1 + \sum_{1}^{\infty}{}_{s}\,(-V)^{s}$$

is a formal expression for $1/\mathfrak{d}$. The rest of the proof, based on (8.4), is almost obvious.

THEOREM (8.2). *Put*

$$\underset{1}{W}{}^{\alpha\beta\gamma}_{\nu\lambda\mu} \overset{\text{def}}{=} \delta^{\alpha\ \beta\gamma}_{\nu(\lambda\mu)} \tag{8.4a}$$

$$\underset{m}{W}{}^{\alpha\beta\gamma}_{\nu\lambda\mu} \overset{\text{def}}{=} \underset{1}{W}{}^{\alpha\beta\gamma}_{\nu\lambda\mu} + \sum_{1}^{m-1}{}_{s}\,{}^{(s)}V^{\alpha\beta\gamma}_{\nu\lambda} \tag{8.4b}$$

$$(m = 2, 3, \ldots)$$

and

$$\underset{m}{U}{}_{\nu\lambda\mu} \overset{\text{def}}{=} H_{\alpha\beta\gamma}\,\underset{m}{W}{}^{\alpha\beta\gamma}_{\nu\lambda\mu} \tag{8.5a}$$

$$\underset{m}{S}{}_{\omega\mu\nu} \overset{\text{def}}{=} \tfrac{1}{2}K_{\omega\mu\nu} - 2\underset{m}{U}{}_{\alpha\nu[\mu}k_{\omega]}{}^{\alpha}. \tag{8.5b}$$

Then

$$U_{\nu\lambda\mu} \overset{2m+2}{=\!=\!=} \underset{m}{U}{}_{\nu\lambda\mu} \tag{8.6a}$$

$$S_{\omega\mu\nu} \overset{2m+3}{=\!=\!=} \underset{m}{S}{}_{\omega\mu\nu}. \tag{8.6b}$$

PROOF. Multiply (8.2) by $\underset{m}{W}{}^{\nu\lambda\mu}_{\xi\eta\zeta}$

$$U_{\alpha\beta\gamma}Z^{\alpha\beta\gamma}_{\nu\lambda\mu}\,\underset{m}{W}{}^{\nu\lambda\mu}_{\xi\eta\zeta} = H_{\nu\lambda\mu}\,\underset{m}{W}{}^{\nu\lambda\mu}_{\xi\eta\zeta}. \tag{8.6c}$$

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[1] See footnote 1 on page 63.

This equation may be simplified. First we have

$$Z^{\alpha\beta\gamma}_{\nu\lambda\mu}\,\underset{m}{W}{}^{\nu\lambda\mu}_{\xi\eta\zeta} =$$

$$= (\delta^{\alpha\ \beta\gamma}_{\nu(\lambda\mu)} - {}^{(1)}V^{\alpha\beta\gamma}_{\nu\lambda\mu})\,(\delta^{\nu\ \lambda\mu}_{\xi(\eta\zeta)} + \sum_{1}^{m-1}{}^{s}\,{}^{(s)}V^{\nu\lambda\mu}_{\xi\eta\zeta}) = \delta^{\alpha\ \beta\gamma}_{\xi(\eta\zeta)} - {}^{(m)}V^{\alpha\beta\gamma}_{\xi\eta\zeta}$$

so that

$$U_{\alpha\beta\gamma}Z^{\alpha\beta\gamma}_{\nu\lambda\mu}\,\underset{m}{W}{}^{\nu\lambda\mu}_{\xi\eta\zeta} = U_{\xi\eta\zeta} - U_{\alpha\beta\gamma}\,{}^{(m)}V^{\alpha\beta\gamma}_{\xi\eta\zeta}.$$

Hence, because of (8.5)a, we have for (8.6)c

$$U_{\nu\lambda\mu} - \underset{m}{U}_{\nu\lambda\mu} = U_{\alpha\beta\gamma}\,{}^{(m)}V^{\alpha\beta\gamma}_{\nu\lambda\mu},$$

and this equation reduces by virtue of (8.3) and (8.1)a,b to (8.6)a. Then, of course, (8.6)b is a consequence of (8.5)b and (8.6)a.

THEOREM (8.3). *Put*

(8.7)a $$\underset{m}{\Gamma}_{\lambda}{}^{\nu}{}_{\mu} \overset{\text{def}}{=\!=} \{\lambda{}^{\nu}{}_{\mu}\} + \underset{m}{S}_{\lambda\mu}{}^{\nu} + \underset{m}{U}{}^{\nu}{}_{\lambda\mu}.$$

Then

(8.7)b $$\Gamma_{\lambda}{}^{\nu}{}_{\mu} \overset{2m+2}{=\!=\!=} \underset{m}{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}.$$

The proof follows at once from (3.1), (8.7)a and (8.6).

COROLLARY (8.1). *We have*

(8.8) $$\Gamma_{\lambda}{}^{\nu}{}_{\mu} \overset{4}{=} \{\lambda{}^{\nu}{}_{\mu}\} + \tfrac{1}{2}K_{\lambda\mu}{}^{\nu} - k_{(\mu}{}^{\alpha}K_{\lambda)}{}^{\nu}{}_{\alpha}$$
$$- [- k_{[\lambda}{}^{\alpha}k_{\mu]}{}^{\beta}K^{\nu}{}_{\alpha\beta} + k^{\nu\beta}k_{[\mu}{}^{\alpha}K_{\lambda]\alpha\beta}].$$

The proof follows from (8.5) and (8.7) for $m = 1$.

NOTE I. Theorem (8.3) shows how fast we can approximate $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. Thus, for instance, the first step ($m = 1$) approximates $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ except for the biquadratic (and higher) products. However, with this degree of accuracy we have

$$k \overset{4}{=} 0,$$

while for the first class $k \neq 0$. Therefore, *in the case of the first class any approximation which disregards the biquadratic* (*and higher*) *products distorts its characteristic property* (*i.e.*, $k \neq 0$). Thus, the lowest applicable step in this case is the second one, $m = 2$, which leads to

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} \overset{6}{=} \underset{2}{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}.$$

The corresponding $\underset{2}{U}$ and $\underset{2}{S}$ have to be taken from (8.5) and (8.4)

(8.9)a $$\underset{2}{U}_{\nu\lambda\mu} = H_{\alpha\beta\gamma}(\delta^{\alpha\beta\gamma}_{\nu(\lambda\mu)} + {}^{(1)}V^{\alpha\beta\gamma}_{\nu\lambda\mu})$$

(8.9)b $$\underset{2}{S}_{\nu\lambda\mu} = \tfrac{1}{2}K_{\nu\lambda\mu} - 2\underset{2}{U}_{\alpha\mu[\lambda}k_{\nu]}{}^{\alpha}.$$

NOTE II. This method of approximation is very handy if one deals with problems connected with $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ without actually knowing its exact form. [1]

This concludes the first part of Chapter II. In the next part we shall deal again with the case $n = 4$. This restriction will enable us to exhibit an explicit formula for $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ in a very simple form.

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[1] Bonnor, W. B., (5); Infeld, L., (1); Schrödinger, E., (5); Udeschini, P., (1), (2), (3), (6), (8); Hittmair, O. — Schrödinger, E., (1); Hoang, P. T., (1); Callaway, J., (1); Kursunoğlu, B., (1), (3); Ikeda, M., (1); Lenoir, M., (2), Narlikar, V. V. — Tiwari, R., (1), (2), (3); Moffat, J., (1), Tonnelat, A., (3), (6), (7), (13); Zanella, A., (1); Maurer — Tison, F., (2). Another method of approximation is dealt with in Hlavatý, V., (7).

CHAPTER II

PART II

The case of $n = 4$

First and second class

9. Auxiliary formulae. In this part of the second chapter we shall discuss the case $n = 4$ and apply the results of the first chapter to the theory developed in the first part of this chapter.[1] In sections 9–13 we shall consider only the case in which the tensor $k_{\lambda\mu}$ belongs to the first class. Then the basic scalars $\underset{1}{\lambda}, \ldots, \underset{4}{\lambda}$ are given by I (6.2), *i.e.*,

$$\alpha \overset{\text{def}}{=} \underset{1}{\lambda} = -\underset{2}{\lambda} = i\sqrt{\sqrt{D} + K} \neq 0, \tag{9.1)a}$$

$$\beta \overset{\text{def}}{=} \underset{3}{\lambda} = -\underset{4}{\lambda} = \sqrt{\sqrt{D} - K} \neq 0. \tag{9.1)b}$$

DEFINITION (9.1). *The scalars* $\underset{ijk}{\lambda}$ *are defined by*

$$\underset{ijk}{\lambda} \overset{\text{def}}{=} 1 + \underset{i}{\lambda}\underset{j}{\lambda} + \underset{j}{\lambda}\underset{k}{\lambda} + \underset{k}{\lambda}\underset{i}{\lambda} \tag{9.2}$$

with the understanding that at least two of the indices i, j, k are different.

The following two Theorems will be proved simultaneously:

THEOREM (9.1)a. *The scalars* $\underset{ijk}{\lambda}$ *are symmetric in their indices and satisfy the conditions*

$$\underset{12i}{\lambda} = 1 + K + \sqrt{D}, \quad \underset{34i}{\lambda} = 1 + K - \sqrt{D} \tag{9.3)a}$$

$$\underset{223}{\lambda} = \underset{114}{\lambda} = \underset{113}{\bar{\lambda}} = \underset{224}{\bar{\lambda}} = 1 - K - \sqrt{D} - 2\sqrt{k} \text{ [2]} \tag{9.3)b}$$

$$\underset{332}{\lambda} = \underset{441}{\lambda} = \underset{331}{\bar{\lambda}} = \underset{442}{\bar{\lambda}} = 1 - K + \sqrt{D} - 2\sqrt{k} \tag{9.3)c}$$

$$\underset{113}{\lambda} = \underset{224}{\lambda} = \underset{223}{\bar{\lambda}} = \underset{114}{\bar{\lambda}} = 1 - K - \sqrt{D} + 2\sqrt{k} \tag{9.3)d}$$

---

[1] This part is based on Hlavatý, V., (18).

[2] The dash over λ denotes as usual the complex conjugate.

(9.3)e $$\underset{331}{\lambda} = \underset{442}{\lambda} = \underset{332}{\bar{\lambda}} = \underset{441}{\bar{\lambda}} = 1 - K + \sqrt{D} + 2\sqrt{k}.$$

THEOREM (9.1)b. *A necessary and sufficient condition that all scalars* $\underset{ijk}{\lambda}$ *be different from zero*

(9.4)a $$\underset{ijk}{\lambda} \neq 0$$

is

(9.4)b $$g \neq 0.$$

This condition is satisfied by virtue of Agreement I (1.1). *In particular,*

(9.5) $$\underset{12i}{\lambda}\ \underset{34j}{\lambda} = g.$$

PROOF. The symmetry of the scalars $\underset{ijk}{\lambda}$ in the indices i, j, k, follows from (9.2). The equations (9.3) follow by simple inspection from I (6.1)a, (9.1) and (9.2). Using (9.3)a,b, I (6.1)a and I (3.3)b one obtains

$$\underset{12i}{\lambda}\ \underset{34j}{\lambda} = (1 + K + \sqrt{D})\,(1 + K - \sqrt{D}) =$$
$$= (1 + K)^2 - D = g.$$

The scalars (9.3)b–e are complex, since $k < 0$. The first statement of Theorem (9.1)b follows from these results.

Let us now consider the tensor X as defined by (3.11), *i.e.*,

(9.6) $$X^{\alpha\beta\gamma}_{\omega\mu\nu} = \delta^{\alpha\beta\,\gamma}_{[\omega\mu]\nu} - 2\delta^{[\alpha}_{\nu} k_{[\omega}{}^{\beta]} k_{\mu]}{}^{\gamma} - 2\delta^{[\alpha}_{[\mu} k_{\omega]}{}^{\beta]} k_{\nu}{}^{\gamma}.$$

In the next Theorem we shall consider the nonholonomic components of this tensor, applying Definition I (8.1):

THEOREM (9.2). *The nonholonomic components* X^{ijk}_{xyz} *of* X *are given by*

(9.7) $$X^{ijk}_{xyz} = \delta^{ij\,k}_{[xy]z} - 2\delta^{[ij]k}_{z[xy]} \underset{xy}{\lambda\lambda} + \delta^{[ij]k}_{xy\,z} \underset{yz}{\lambda\lambda} - \delta^{[ij]k}_{yx\,z} \underset{xz}{\lambda\lambda}.$$ [1]

They are invariant with respect to the transformations I (9.5).

---

[1] We remind the reader that the indices x, y, z, t do not follow the usual Einstein summation convention.

PROOF. The nonholonomic components may be derived from (9.6)

$$X^{ijk}_{xyz} = \delta^{\,i\,j\,k}_{[xy]z} - 2\delta^{[i}_{z}k_{[x}{}^{j]}k_{y]}{}^{k} - 2\delta^{[i}_{[y}k_{x]}{}^{j]}k_{z}{}^{k}. \tag{9.8}$$

Their invariance follows from (9.8) and from Theorem I (9.3). Substituting into (9.8) from I (9.1)a, *i.e.*,

$$k_x{}^i = \underset{x}{\lambda}\delta^i_x, \tag{9.9}$$

one obtains (9.7).

NOTE. The equation (9.7) shows that X^{ijk}_{xyz} vanishes whenever ijk is not a permutation of xyz. The remaining components may be described by the following two sets of formulae: If $y = z$, $x \neq y$ we have from (9.7)

$$X^{xyy}_{xyy} = -X^{yxy}_{xyy} = \tfrac{1}{2}\underset{xyy}{\lambda}\ . \tag{9.10)a}$$

(Note that $X^{yyx}_{xyy} = 0$.)

If, however, xyz are three different indices, then the corresponding components X^{ijk}_{xyz} may be taken from the first row of the following self-explanatory table

$$\begin{array}{cccc}
 & X^{xyz}_{\cdot\cdot\cdot} & X^{yzx}_{\cdot\cdot\cdot} & X^{zxy}_{\cdot\cdot\cdot} \\
X^{\cdot\cdot\cdot}_{xyz} & \tfrac{1}{2}\Bigl(\underset{xyz}{\lambda} - \underset{xy}{\lambda\lambda}\Bigr) & -\tfrac{1}{2}\underset{xy}{\lambda\lambda} & -\tfrac{1}{2}\underset{xy}{\lambda\lambda} \\
X^{\cdot\cdot\cdot}_{yzx} & -\tfrac{1}{2}\underset{yz}{\lambda\lambda} & \tfrac{1}{2}\Bigl(\underset{xyz}{\lambda} - \underset{yz}{\lambda\lambda}\Bigr) & -\tfrac{1}{2}\underset{yz}{\lambda\lambda} \\
X^{\cdot\cdot\cdot}_{zxy} & -\tfrac{1}{2}\underset{zx}{\lambda\lambda} & -\tfrac{1}{2}\underset{zx}{\lambda\lambda} & \tfrac{1}{2}\Bigl(\underset{xyz}{\lambda} - \underset{zx}{\lambda\lambda}\Bigr).
\end{array} \tag{9.10)b}$$

The remaining rows may be obtained by suitable permutation of indices. The formulae (9.10) will be very useful for later purposes.

Denote by $\underset{xyz}{\varDelta}/8$ the determinant of the matrix of the table (9.10)b.

$$\underset{xyz}{\varDelta} \overset{\text{def}}{=} \begin{vmatrix}
\underset{xyz}{\lambda} - \underset{xy}{\lambda\lambda} & -\underset{xy}{\lambda\lambda} & -\underset{xy}{\lambda\lambda} \\
-\underset{yz}{\lambda\lambda} & \underset{xyz}{\lambda} - \underset{yz}{\lambda\lambda} & -\underset{yz}{\lambda\lambda} \\
-\underset{zx}{\lambda\lambda} & -\underset{zx}{\lambda\lambda} & \underset{xyz}{\lambda} - \underset{zx}{\lambda\lambda}
\end{vmatrix}. \tag{9.11}$$

THEOREM (9.3). *We have*

$$\underset{xyz}{\varDelta} = \Bigl(\underset{xyz}{\lambda}\Bigr)^2 \neq 0. \tag{9.12}$$

PROOF. One obtains the equation (9.12) by applying in turn the following elementary operations to the determinant (9.11): 1) adding the last two rows to the first one, 2) subtracting the second column from the third one, 3) adding the second row to the third one, and finally, 4) subtracting the first column from the second one. The inequality (9.12) follows from Theorem (9.1)b.

10. The tensors $Z^{\alpha\beta\gamma}_{\omega\mu\nu}$[1] and $L_{\omega\mu\nu}$.

In this section we shall use the tensors $\overset{xy}{Q}_{\omega\mu}$ and $\underset{x}{E}_{\lambda}{}^{\nu}$ defined by I (10.6)b and I (10.10). Although these tensors have been constructed without the help of basic vectors we have according to I (10.8) and I (10.11)

$$(10.1)\text{a} \qquad \overset{xy}{Q}_{\omega\mu} = 2\overset{x}{a}_{[\omega}\overset{y}{a}_{\mu]}$$

$$(10.1)\text{b} \qquad \overset{xy}{Q}{}^{\omega\mu} = 2\underset{y}{a}{}^{[\omega}\underset{x}{a}{}^{\mu]} \qquad (xy = 12,\ 34)$$

$$(10.1)\text{c} \qquad \underset{x}{E}_{\lambda}{}^{\nu} = \overset{x}{a}_{\lambda}\underset{x}{a}{}^{\nu} \qquad (x = 1, \ldots, 4),$$

so that, in particular,

$$(10.2) \qquad \underset{x}{E}_{[\omega}{}^{[\alpha}\underset{y}{E}_{\mu]}{}^{\beta]} = \underset{y}{E}_{[\omega}{}^{[\alpha}\underset{x}{E}_{\mu]}{}^{\beta]} = -\tfrac{1}{4}\overset{xy}{Q}_{\omega\mu}\overset{xy}{Q}{}^{\alpha\beta}$$

In the next Theorem we denote by $RT^{\cdots}_{\cdots}$ the real part of a complex tensor $T^{\cdots}_{\cdots}$.

THEOREM (10.1)a. *Put*

$$(10.3)\text{a} \qquad Z^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\text{def}}{=} \sum{}^{xyz}\frac{1}{\underset{xyz}{\lambda}}\underset{x}{E}_{[\omega}{}^{[\alpha}\underset{y}{E}_{\mu]}{}^{\beta]}\underset{z}{E}_{\nu}{}^{\gamma}$$

Then

$$(10.3)\text{b} \qquad \tfrac{1}{2}Z^{\alpha\beta\gamma}_{\omega\mu\nu} =$$

$$= \frac{1}{1+K+\sqrt{D}}\Big(-\tfrac{1}{4}\overset{12}{Q}_{\omega\mu}\overset{12}{Q}{}^{\alpha\beta}\delta^{\gamma}_{\nu} + 2R\overset{4}{\underset{3}{\sum}}{}^{z}\,\underset{1}{E}_{[\omega}{}^{[\alpha}\underset{z}{E}_{\mu]}{}^{\beta]}\underset{2}{E}_{\nu}{}^{\gamma}\Big) +$$

$$+ \frac{1}{1+K-\sqrt{D}}\Big(-\tfrac{1}{4}\overset{34}{Q}_{\omega\mu}\overset{34}{Q}{}^{\alpha\beta}\delta^{\gamma}_{\nu} + 4R\underset{(4}{E}_{\nu}{}^{\gamma}\underset{3)}{E}_{[\omega}{}^{[\alpha}\underset{1}{E}_{\mu]}{}^{\beta]}\Big) +$$

---

[1] This tensor is not connected with the tensor of the same name defined by (8.1)c.

$$+ R\,\frac{1}{1-K-\sqrt{D}-2\sqrt{k}}\,(2\underset{2}{E}_{[\omega}{}^{[\alpha}\underset{3}{E}_{\mu]}{}^{\beta]}\underset{2}{E}_{\nu}{}^{\gamma} + 2\underset{1}{E}_{[\omega}{}^{[\alpha}\underset{4}{E}_{\mu]}{}^{\beta]}\underset{1}{E}_{\nu}{}^{\gamma}) +$$

$$+ R\,\frac{1}{1-K+\sqrt{D}-2\sqrt{k}}\,(2\underset{2}{E}_{[\omega}{}^{[\alpha}\underset{3}{E}_{\mu]}{}^{\beta]}\underset{3}{E}_{\nu}{}^{\gamma} + 2\underset{1}{E}_{[\omega}{}^{[\alpha}\underset{4}{E}_{\mu]}{}^{\beta]}\underset{4}{E}_{\nu}{}^{\gamma})$$

where the tensors Q and E are to be taken from I(10.6)b *and* I (10.10) (*so that the right hand side of* (10.3)b *does not explicitly contain the basic vectors*).

PROOF. Let us consider the set of all tensors on the right hand side of (10.3)a which have the coefficient $\frac{1}{\underset{233}{\lambda}}$ and $\frac{1}{\underset{144}{\lambda}} = \frac{1}{\underset{233}{\lambda}}$

(10.4)a
$$\frac{1}{\underset{233}{\lambda}}\,(\underset{2}{E}_{[\omega}{}^{[\alpha}\underset{3}{E}_{\mu]}{}^{\beta]} + \underset{3}{E}_{[\omega}{}^{[\alpha}\underset{2}{E}_{\mu]}{}^{\beta]})\,\underset{3}{E}_{\nu}{}^{\gamma} +$$
$$\frac{1}{\underset{144}{\lambda}}\,(\underset{1}{E}_{[\omega}{}^{[\alpha}\underset{4}{E}_{\mu]}{}^{\beta]} + \underset{4}{E}_{[\omega}{}^{[\alpha}\underset{1}{E}_{\mu]}{}^{\beta]})\underset{4}{E}_{\nu}{}^{\gamma} =$$
$$= \frac{2}{\underset{233}{\lambda}}\,(\underset{2}{E}_{[\omega}{}^{[\alpha}\underset{3}{E}_{\mu]}{}^{\beta]}\underset{3}{E}_{\nu}{}^{\gamma} + \underset{1}{E}_{[\omega}{}^{[\alpha}\underset{4}{E}_{\mu]}{}^{\beta]}\underset{4}{E}_{\nu}{}^{\gamma}).$$

Next, consider all tensors with the coefficients $\frac{1}{\underset{331}{\lambda}}$ and $\frac{1}{\underset{442}{\lambda}} = \frac{1}{\underset{331}{\lambda}}$, *i.e.*,

(10.4)b
$$\frac{1}{\underset{331}{\lambda}}\,(\underset{1}{E}_{[\omega}{}^{[\alpha}\underset{3}{E}_{\mu]}{}^{\beta]} + \underset{3}{E}_{[\omega}{}^{[\alpha}\underset{1}{E}_{\mu]}{}^{\beta]})\underset{3}{E}_{\nu}{}^{\gamma} +$$
$$+ \frac{1}{\underset{442}{\lambda}}\,(\underset{2}{E}_{[\omega}{}^{[\alpha}\underset{4}{E}_{\mu]}{}^{\beta]} + \underset{4}{E}_{[\omega}{}^{[\alpha}\underset{2}{E}_{\mu]}{}^{\beta]})\underset{4}{E}_{\nu}{}^{\gamma} =$$
$$= \frac{2}{\underset{331}{\lambda}}\,(\underset{1}{E}_{[\omega}{}^{[\alpha}\underset{3}{E}_{\mu]}{}^{\beta]}\underset{3}{E}_{\nu}{}^{\gamma} + \underset{2}{E}_{[\omega}{}^{[\alpha}\underset{4}{E}_{\mu]}{}^{\beta]}\underset{4}{E}_{\nu}{}^{\gamma}).$$

The scalars $\underset{233}{\lambda}$ and $\underset{133}{\lambda}$ are complex-conjugate and the same is true for the tensors $\underset{1}{E}_{\lambda}{}^{\nu}$ and $\underset{2}{E}_{\lambda}{}^{\nu}$. Hence, adding the tensors (10.4)a,b we obtain according to (9.3)

$$4R(\underset{2}{E}{}_{[\omega}{}^{[\alpha}\underset{3}{E}{}_{\mu]}{}^{\beta]}\underset{3}{E}{}_{\nu}{}^{\gamma} + \underset{1}{E}{}_{[\omega}{}^{[\alpha}\underset{4}{E}{}_{\mu]}{}^{\beta]}\underset{4}{E}{}_{\nu}{}^{\gamma})\,\frac{1}{1+\sqrt{D}-K-2\sqrt{k}}$$

which is twice the tensor in the last row of (10.3)b. The remaining rows may be obtained in a similar manner. The remaining part of the Theorem is obvious.

The components $Z_{\omega\mu\nu}^{\alpha\beta\gamma}$ given by (10.3)b are rather complicated. However, the corresponding nonholonomic components are extremely simple.

THEOREM (10.1)b. *The nonholonomic components of the tensor Z given by* (10.3)b *are*

(10.3)c
$$Z_{xyz}^{ijk} = \frac{1}{\underset{xyz}{\lambda}}\,\delta_{[xy]z}^{[ij]k}.$$

PROOF. The nonholonomic components of $\underset{x}{E}{}_{\lambda}{}^{\nu}$ are

$$\underset{x}{E}{}_j{}^i = \underset{j}{a}{}^{\lambda}\overset{i}{a}{}_{\nu}\underset{x}{E}{}_{\lambda}{}^{\nu} = (\underset{j}{a}{}^{\lambda}\overset{x}{a}{}_{\lambda})\,(\overset{i}{a}{}_{\nu}\underset{x}{a}{}^{\nu}) = \delta_j^x\delta_x^i.$$

Therefore, according to (10.3)a [1]

(10.5)
$$Z_{xyz}^{ijk} = \sum\nolimits^{x'y'z'} \frac{1}{\underset{x'y'z'}{\lambda}}\,\underset{x'}{E}{}_{[x}{}^{[i}\underset{y'}{E}{}_{y]}{}^{j]}\underset{z'}{E}{}_z{}^k$$

$$= \sum\nolimits^{x'y'z'} \frac{1}{\underset{x'y'z'}{\lambda}}\,\delta_{[x\;y]\,x'y'z\,z'}^{x'y'[i\;\,j]z'k} = \frac{1}{\underset{xyz}{\lambda}}\,\delta_{[xy]z}^{[ij]k}.$$

THEOREM (10.2). *Let*

(10.6)a
$$L_{\omega\mu\nu} = L_{[\omega\mu]\nu}$$

be a tensor whose nonholonomic components are

(10.6)b
$$L_{xyz} \overset{\text{def}}{=} \frac{K_{xyz}}{\underset{xyz}{\lambda}}.\,{}^{2}$$

---

[1] The indices x', y', z', introduced in (10.5), do not follow the summation convention.

[2] K_{xyz} are nonholonomic components of the tensor $K_{\omega\nu\mu}$ defined by (3.5).

Then

(10.6)c
$$L_{\omega\mu\nu} = Z^{\alpha\beta\gamma}_{\omega\mu\nu} K_{\alpha\beta\gamma},$$

where $Z^{\alpha\beta\gamma}_{\omega\mu\nu}$ *is defined by* (10.3)b.

PROOF. The equation (10.6)b may be rewritten as follows:

(10.7)
$$L_{xyz} = \frac{1}{\underset{xyz}{\lambda}} \underset{x}{a}^{[\alpha} \underset{y}{a}^{\beta]} \underset{z}{a}^{\gamma} K_{\alpha\beta\gamma}.$$

Thus, by virtue of (10.7), (10.1)c and (10.3)a

(10.8)
$$L_{\omega\mu\nu} = \sum^{xyz} L_{xyz} \overset{x}{a}_{[\omega} \overset{y}{a}_{\mu]} \overset{z}{a}_{\nu} =$$
$$= \sum^{xyz} \frac{1}{\underset{xyz}{\lambda}} \underset{x}{a}^{[\alpha} \underset{y}{a}^{\beta]} \underset{z}{a}^{\gamma} \overset{x}{a}_{[\omega} \overset{y}{a}_{\mu]} \overset{z}{a}_{\nu} K_{\alpha\beta\gamma} =$$
$$= \sum^{xyz} \frac{1}{\underset{xyz}{\lambda}} \underset{x}{E}_{[\omega}{}^{[\alpha} \underset{y}{E}_{\mu]}{}^{\beta]} \underset{z}{E}_{\nu}{}^{\gamma} K_{\alpha\beta\gamma} = Z^{\alpha\beta\gamma}_{\omega\mu\nu} K_{\alpha\beta\gamma}.$$

The tensor $L_{\omega\mu\nu}$ will play an important role in the next section.

11. The tensors $S_{\omega\mu}{}^{\nu}$ and $U^{\nu}{}_{\lambda\mu}$. In this section we shall consider the solution $S_{\omega\mu\nu}$ of (3.12). The following two Theorems will be proved simultaneously.

THEOREM (11.1). *A necessary and sufficient condition for the existence and uniqueness of the solution* $S_{\omega\mu\nu}$ *of* (3.12) *is* (9.4)b.

THEOREM (11.2). *If* (9.4)b *is satisfied, then the system* (3.12) *admits a unique solution*

(11.1)
$$2S_{\omega\mu\nu} = L_{\omega\mu\nu} + 3L_{[\alpha\beta\nu]} k_{\omega}{}^{\alpha} k_{\mu}{}^{\beta}$$

where the tensor $L_{\omega\mu\nu}$ *is defined by* (10.6)c.

PROOF. Rewrite the system (3.12) for nonholonomic components

(11.2)
$$S_{ijk} X^{ijk}_{pqr} = \tfrac{1}{2} K_{pqr}.$$

Assume first that pqr are three different indices, and put in turn

$$pqr = xyz; \quad pqr = yzx, \quad pqr = zxy.$$

Then we obtain from (11.2) and (9.10)

$$
\begin{aligned}
(11.3)\qquad & S_{ijk}X^{ijk}_{xyz} = \\
& S_{xyz}\Big(\underset{xyz}{\lambda} - \underset{xy}{\lambda\lambda}\Big) - S_{yzx}\underset{xy}{\lambda\lambda} - S_{zxy}\underset{xy}{\lambda\lambda} = \tfrac{1}{2}K_{xyz} \\
& S_{ijk}X^{ijk}_{yzx} = \\
& - S_{xyz}\underset{yz}{\lambda\lambda} + S_{yzx}\Big(\underset{xyz}{\lambda} - \underset{yz}{\lambda\lambda}\Big) - S_{zxy}\underset{yz}{\lambda\lambda} = \tfrac{1}{2}K_{yzx} \\
& S_{ijk}X^{ijk}_{zxy} = \\
& - S_{xyz}\underset{zx}{\lambda\lambda} - S_{yzx}\underset{zx}{\lambda\lambda} + S_{zxy}\Big(\underset{xyz}{\lambda} - \underset{zx}{\lambda\lambda}\Big) = \tfrac{1}{2}K_{zxy}.
\end{aligned}
$$

The system (11.3) represents three linear non-homogeneous equations for three unknowns S_{xyz}, S_{yzx}, and S_{zxy}. It has a unique solution if and only if its determinant $\underset{xyz}{\Delta} \neq 0$ or, according to Theorems (9.3) and (9.1)b, if and only if (9.4)b holds. If this condition is satisfied, then we obtain from (11.3), using (9.12),

$$
(11.4)\qquad 2S_{xyz} = \frac{1}{\Big(\underset{xyz}{\lambda}\Big)^2}
\begin{vmatrix}
K_{xyz} & -\underset{xy}{\lambda\lambda} & -\underset{xy}{\lambda\lambda} \\
K_{yzx} & \underset{xyz}{\lambda} - \underset{yz}{\lambda\lambda} & -\underset{yz}{\lambda\lambda} \\
K_{zxy} & -\underset{zx}{\lambda\lambda} & \underset{xyz}{\lambda} - \underset{zx}{\lambda\lambda}
\end{vmatrix} =
$$

$$
= \frac{1}{\underset{xyz}{\lambda}}\big[3K_{[xyz]}(1 + \underset{xy}{\lambda\lambda}) - 2K_{z[xy]}\big].\text{[1]}
$$

On the other hand,

$$
3K_{[xyz]} - 2K_{z[xy]} = K_{xyz}
$$

so that

$$
(11.5)\text{a}\qquad 2S_{xyz} = \frac{1}{\underset{xyz}{\lambda}}\big[K_{xyz} + 3K_{[xyz]}\underset{xy}{\lambda\lambda}\big].
$$

Assume now $p \neq q = r$ in (11.2), and put

$$
p = x, \quad q = r = y \neq x.
$$

---

[1] One easily obtains this expression by applying the following elementary operations to the determinant involved in (11.4): 1) adding the last two rows to the first one, 2) adding the third row to the second one, 3) subtracting the second column from the last one.

Then we have by virtue of (11.2) and (9.10)a

$$2S_{ijk}X^{i\,jk}_{xyy} = S_{xyy}\underset{xyy}{\lambda} - S_{yxy}\underset{xyy}{\lambda} = 2S_{xyy}\underset{xyy}{\lambda} = K_{xyy}$$

or

(11.5)b $$2S_{xyy} = \frac{1}{\underset{xyy}{\lambda}} K_{xyy}.$$

If we put $z = y$ in (11.5)a, we obtain (11.5)b. Hence, we may disregard (11.5)b, provided that we include the possibility $z = y$ in (11.5)a. This latter equation may be rewritten by means of (10.6)b

(11.6)a $$2S_{xyz} = L_{xyz} + 3L_{[xyz]}\underset{x}{\lambda}\underset{y}{\lambda}$$

or by virtue of (9.9)

(11.6)b $$2S_{xyz} = L_{xyz} + L_{[ijz]}k_x{}^i k_y{}^j.$$

This equation is obviously equivalent to (11.1).

NOTE. Using the matrix calculus, Bose, S. N., (4) found a formula which may be easily converted into (11.1). (*Cf.* also Bose, S. N., (5).)

THEOREM (11.3). *The tensor* $U^\nu{}_{\lambda\mu}$ *defined by* (3.2) *is expressible in terms of* $g_{\lambda\mu}$ *as follows:*

(11.7) $$U^\nu{}_{\lambda\mu} = -\,h^{\omega\nu}k_{(\lambda}{}^{\gamma}L_{\mu)\omega\gamma}.$$

PROOF. Substituting into (3.2) for S from (11.1) we find

(11.8) $$\begin{aligned} U^\nu{}_{\lambda\mu} &= h^{\omega\nu}(S_{\omega\mu\gamma}k_\lambda{}^\gamma + S_{\omega\lambda\gamma}k_\mu{}^\gamma) \\ &= \tfrac{1}{2}h^{\omega\nu}(L_{\omega\mu\gamma}k_\lambda{}^\gamma + L_{\omega\lambda\gamma}k_\mu{}^\gamma + 3L_{[\alpha\beta\gamma]}k_\omega{}^\alpha k_\mu{}^\beta k_\lambda{}^\gamma \\ &\quad + 3L_{[\alpha\beta\gamma]}k_\omega{}^\alpha k_\lambda{}^\beta k_\mu{}^\gamma) = -\,\tfrac{1}{2}h^{\omega\nu}(k_\lambda{}^\gamma L_{\mu\omega\gamma} + k_\mu{}^\gamma L_{\lambda\omega\gamma}) \\ &= -\,h^{\omega\nu}k_{(\lambda}{}^\gamma L_{\mu)\omega\gamma}. \end{aligned}$$

Lowering the index ν in (11.7) we obtain, using (10.6)b, an equation whose nonholonomic expression is

$$U_{zxy} = \frac{1}{2\underset{xyz}{\lambda}}(K_{zyj}k_x{}^j + K_{zxj}k_y{}^j) = -\,\frac{1}{\underset{xyz}{\lambda}}k_{(x}{}^j K_{y)zj}.$$

Hence, a necessary and sufficient condition for

$$U_{\lambda\mu\nu} = 0$$

is

$$k_{(\lambda}{}^{\alpha} K_{\mu)\nu\alpha} = 0$$

as has been stated in Corollary (7.1).

We conclude this section with a Theorem concerning the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$.

THEOREM (11.4). *A necessary and sufficient condition for the system* (2.1) *to admit a unique solution* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ *is* (9.4)b. *If this condition is satisfied, then*

$$(11.9) \quad \Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{\lambda{}^{\nu}{}_{\mu}\} + h^{\alpha\nu}(\tfrac{1}{2}L_{\lambda\mu\alpha} + \tfrac{3}{2}L_{[\beta\gamma\alpha]}k_{\lambda}{}^{\beta}k_{\mu}{}^{\gamma} - k_{(\lambda}{}^{\gamma}L_{\mu)\alpha\gamma})$$

where L is defined by (10.6).

The proof follows easily from Theorems (3.1), (11.1), (11.2) and (11.3).[1] [2]

NOTE. Comparing this Theorem with Theorem (5.2) we shall find in the next section the tensor Y involved in (5.9).

12. The tensor $Y^{\alpha\beta\gamma}_{\omega\mu\nu}$. In this section we shall find the explicit form of the tensor Y defined in Theorem (5.1), and involved in the equation (5.10), *i.e.*,

$$(12.1) \qquad 2S_{\omega\mu\nu} = Y^{\alpha\beta\gamma}_{\omega\mu\nu} K_{\alpha\beta\gamma}.$$

THEOREM (12.1). *Put*

$$(12.2)\text{a} \qquad {}^{0}Y^{\alpha\beta\gamma}_{\omega\mu\nu} \stackrel{\text{def}}{=} Z^{\alpha\beta\gamma}_{\omega\mu\nu} + 3Z^{\alpha\beta\gamma}_{[\xi\eta\nu]} k_{\omega}{}^{\xi} k_{\mu}{}^{\eta}$$

where Z is defined by (10.3). *Then*

$$(12.3) \qquad 2S_{\omega\mu\nu} = {}^{0}Y^{\alpha\beta\gamma}_{\omega\mu\nu} K_{\alpha\beta\gamma}.$$

PROOF. One easily obtains (12.3) from (12.2)a, (11.1) and (10.6)c.

NOTE. Comparing (12.1) and (12.3) one finds

$$(12.4)\text{a} \qquad Y^{\alpha\beta\gamma}_{\omega\mu\nu} = {}^{0}Y^{\alpha\beta\gamma}_{\omega\mu\nu} + T^{\alpha\beta\gamma}_{\omega\mu\nu}$$

where the unknown tensor T must satisfy the condition

$$(12.4)\text{b} \qquad K_{\alpha\beta\gamma} T^{\alpha\beta\gamma}_{\omega\mu\nu} = 0.$$

In order to find T we need the following

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[1] The method used here for $n = 4$ has been generalized (for all three classes) for any n in Wrede, R., (1).

[2] See also Kichenassamy, S., (1).

THEOREM (12.2). *The nonholonomic components of the tensor* ${}^0Y$ *are*

$$(12.2)\text{b} \qquad {}^0Y^{ijk}_{xyz} = \frac{1}{\underset{xyz}{\lambda}}\,[\delta^{ijk}_{[xy]z} + 3\delta^{[ij]k}_{[xyz]}\underset{x}{\lambda}\underset{y}{\lambda}].$$

PROOF. The nonholonomic form of (12.2)a is by (9.9)

$$(12.5) \qquad {}^0Y^{ijk}_{xyz} = Z^{ijk}_{xyz} + 3Z^{ijk}_{[xyz]}\underset{x}{\lambda}\underset{y}{\lambda}.$$

Substituting into this equation from (10.3)c one obtains (12.2)b.

NOTE. As in the case of X the non-vanishing components ${}^0Y^{ijk}_{xyz}$ may be described by two tables: If $y = z$, then we have from (12.2)b

$$(12.2)\text{c} \qquad {}^0Y^{xyy}_{xyy} = -\,{}^0Y^{yxy}_{xyy} = \frac{1}{2\underset{xyy}{\lambda}}$$

$$\text{(note that } {}^0Y^{yyx}_{xyy} = 0\text{)}$$

while for three different indices x, y, z we find according to (12.2)b the following self-explanatory table $\left(\text{where } \rho \overset{\text{def}}{=} \frac{1}{2\underset{xyz}{\lambda}}\right)$

(12.2)d

	${}^0Y^{xyz}_{\cdots}$	${}^0Y^{yzx}_{\cdots}$	${}^0Y^{zxy}_{\cdots}$
${}^0Y^{\cdots}_{xyz}$	$\rho(1 + \underset{x}{\lambda}\underset{y}{\lambda})$	$\rho\underset{x}{\lambda}\underset{y}{\lambda}$	$\rho\underset{x}{\lambda}\underset{y}{\lambda}$
${}^0Y^{\cdots}_{yzx}$	$\rho\underset{y}{\lambda}\underset{z}{\lambda}$	$\rho(1 + \underset{y}{\lambda}\underset{z}{\lambda})$	$\rho\underset{y}{\lambda}\underset{z}{\lambda}$
${}^0Y^{\cdots}_{zxy}$	$\rho\underset{z}{\lambda}\underset{x}{\lambda}$	$\rho\underset{z}{\lambda}\underset{x}{\lambda}$	$\rho(1 + \underset{z}{\lambda}\underset{x}{\lambda})$

THEOREM (12.3). *The tensor* ${}^0Y^{\alpha\beta\gamma}_{\omega\mu\nu}$ *defined by* (12.2)a *is identical with the tensor* $Y^{\alpha\beta\gamma}_{\omega\mu\nu}$ *defined in Theorem* (5.1),

$$(12.6) \qquad Y^{\alpha\beta\gamma}_{\omega\mu\nu} = Z^{\alpha\beta\gamma}_{\omega\mu\nu} + 3Z^{\alpha\beta\gamma}_{[\xi\eta\nu]}k_{\omega}{}^{\xi}k_{\mu}{}^{\eta}.$$

PROOF. Using (12.2)b and (9.8) (or (12.2)cd and (9.10)) one finds

$$(12.7)\text{a} \qquad X^{abc}_{ijk}\,{}^0Y^{ijk}_{pqr} = \delta^{[ab]c}_{[pq]r}$$

or in holonomic components

$$X^{\alpha\beta\gamma}_{\xi\eta\zeta}\,{}^0Y^{\xi\eta\zeta}_{\omega\mu\nu} = \delta^{[\alpha\beta]\gamma}_{[\omega\mu]\nu}. \tag{12.7)b}$$

Comparing Theorems (5.2) and (11.4) we see that the condition (9.4)b is equivalent to the requirement that the matrix $((X))$ be of rank $m = 24$. If this condition is satisfied, then the system (12.7)b admits only one tensor ${}^0Y$. The same is true for the tensor Y involved in (5.4). Hence, $Y = {}^0Y$.

NOTE. According to this Theorem we have $T = 0$ for the tensor T involved in (12.4).

13. The components $\Gamma_j{}^i{}_k$.

LEMMA (13.1)a. *The nonholonomic components* $\{{}_j{}^i{}_k\}$ *of the Christoffel symbols* $\{{}_\lambda{}^\nu{}_\mu\}$ *are*

$$\{{}_j{}^i{}_k\} = -\underset{j}{a^\lambda}\,\underset{k}{a^\mu}\nabla_\mu \overset{i}{a}_\lambda \tag{13.1)a}$$

where the a's are basic vectors.

PROOF. The relationship of $\{{}_\lambda{}^\nu{}_\mu\}$ and $\{{}_j{}^i{}_k\}$ is given by the formula

$$\{{}_j{}^i{}_k\} = A^i_\nu[A^\lambda_j A^\mu_k\{{}_\lambda{}^\nu{}_\mu\} + A^\mu_k\partial_\mu A^\nu_j]. \tag{13.2}$$

This equation reduces by virtue of I(8.1) and I (8.2) to

$$\{{}_j{}^i{}_k\} = \overset{i}{a}_\nu\,\underset{k}{a^\mu}\nabla_\mu \underset{j}{a^\nu} = -\underset{j}{a^\lambda}\,\underset{k}{a^\mu}\nabla_\mu \overset{i}{a}_\lambda.$$

LEMMA (13.1)b. *We also have*

$$\{{}_x{}^t{}_y\} = -\underset{x}{\rho}\,\underset{y}{\rho}\,\underset{x}{u^\alpha}\,\underset{y}{u^\beta}\,\underset{x}{E_\alpha{}^\lambda}\,\underset{y}{E_\beta{}^\mu}\,\nabla_\mu\,\overset{t}{\rho}\,\overset{t}{u}_\gamma\,\underset{t}{E}_\lambda{}^\gamma \tag{13.1)b}$$

where the ρ's and the u's are taken from Theorem I (10.3), *and the E's are defined by* I (10.10).

The proof is obvious.

THEOREM (13.1). *The nonholonomic components* $\Gamma_x{}^t{}_y$ *of* $\Gamma_\lambda{}^\nu{}_\mu$ *are given by*

$$\Gamma_x{}^t{}_y = \{{}_x{}^t{}_y\} + \tag{13.3}$$

$$+ \tfrac{1}{2}\sum\nolimits^z h^{zt}\,\frac{1}{\underset{xyz}{\lambda}}\,[K_{xyz} + 3K_{[tjz]}k_x{}^i k_y{}^j - 2k_{(x}{}^i K_{y)zi}]$$

where the h^{zt} *are to be taken from* I (8.5). *The construction of* $\Gamma_x{}^t{}_y$ *does not require the actual knowledge of the nonholonomic frame.*

PROOF. One obtains (13.3) using the nonholonomic form of (11.9) and (10.6)b. According to the previous Lemma the construction of $\{{}_x{}^t{}_y\}$ does not require the actual knowledge of the nonholonomic frame. The same is true of the components K_{xyz} because

$$K_{xyz} = \underset{x}{\rho}\,\underset{y}{\rho}\,\underset{z}{\rho}\,\underset{x}{u^{\alpha}}\,\underset{y}{u^{\beta}}\,\underset{z}{u^{\gamma}}\,\underset{x}{E_{\alpha}{}^{\omega}}\,\underset{y}{E_{\beta}{}^{\mu}}\,\underset{z}{E_{\gamma}{}^{\nu}}\,K_{\omega\mu\nu}.$$

14. Second class. In this section we shall consider the tensor $k_{\lambda\mu}$ to be of the second class so that the basic scalars are given by I (7.2) and I (7.3), *i.e.*,

(14.1)a $$\alpha \overset{\text{def}}{=} \underset{1}{\lambda} = -\underset{2}{\lambda} = \sqrt{-2K}$$

for $K > 0$,

(14.1)b $$\beta \overset{\text{def}}{=} \underset{3}{\lambda} = -\underset{4}{\lambda} = 0$$

(14.2)a $$\alpha \overset{\text{def}}{=} \underset{1}{\lambda} = -\underset{2}{\lambda} = 0$$

for $K < 0$.

(14.2)b $$\beta \overset{\text{def}}{=} \underset{3}{\lambda} = -\underset{4}{\lambda} = \sqrt{-2K}$$

The following two Theorems will be proved simultaneously.

THEOREM (14.1)a. *The symmetric scalars* $\underset{ijk}{\lambda}$ *defined by* (9.2) *satisfy the following equations: If* $K > 0$, *then*

(14.3)a $$\underset{12i}{\lambda} = g$$

(14.3)b $$\underset{34i}{\lambda} = 1$$

(14.3)c $$\underset{223}{\lambda} = \underset{114}{\lambda} = \underset{113}{\lambda} = \underset{224}{\lambda} = 2 - g$$

(14.3)d $$\underset{233}{\lambda} = \underset{144}{\lambda} = \underset{313}{\lambda} = \underset{442}{\lambda} = 1;$$

if $K < 0$, *then*

(14.4)a $$\underset{12i}{\lambda} = 1$$

(14.4)b $$\underset{34i}{\lambda} = g$$

(14.4)c $$\underset{223}{\lambda} = \underset{114}{\lambda} = \underset{113}{\lambda} = \underset{224}{\lambda} = 1$$

(14.4)d $$\underset{233}{\lambda} = \underset{144}{\lambda} = \underset{331}{\lambda} = \underset{442}{\lambda} = 2 - g.$$

THEOREM (14.1)b. *A necessary and sufficient condition that all the scalars* $\underset{ijk}{\lambda}$ *be different from zero*

(14.5)a $$\underset{ijk}{\lambda} \neq 0$$

is

(14.5)b $$g(g-2) \neq 0.$$

In particular,

$$\underset{12i}{\lambda}\;\underset{34j}{\lambda} = g.$$ [1]

PROOF. For the second class we have, by I (1.6)b, I (3.3)b,

(14.6)a $$k = 0$$

(14.6)b $$K \neq 0$$

(14.6)c $$g = 1 + 2K.$$

Substituting from (14.6) and I (7.1) into (9.3) one obtains (14.3) and (14.4). Then Theorem (14.1)b is obvious.

AGREEMENT (14.1). *Throughout this section we shall assume that* (14.5)b *is satisfied.*

THEOREM (14.2). *The following statements, proved for the first class, also hold for the second class, i.e., for* (14.3) *and* (14.4)*: Theorems* (9.2), (9.3), (10.1)a,b, (10.2), (12.1), (12.2), (12.3), (13.1) *and Lemmas* (13.1)a,b.

PROOF. In none of these statements did we use the fact $k \neq 0$, characteristic for the first class. Hence, they also hold for $k = 0$, which characterizes (together with $K \neq 0$) the second class.

The following three Theorems which replace Theorems (11.1), (11.2), (11.3) and (11.4) will be proved simultaneously.

THEOREM (14.3). *A necessary and sufficient condition for the existence and uniqueness of the solution* $S_{\omega\mu\nu}$ *of* (3.12) *is* (14.5)b.

THEOREM (14.4). *If* (14.4)b *is satisfied, then the system* (3.12) *admits a unique solution given by* (11.1) *and* $U^{\nu}{}_{\lambda\mu}$ *is given by* (11.7).

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[1] See also Kichenassamy, S., (1).

THEOREM (14.5). *A necessary and sufficient condition for the system* (2.1) *to admit a unique solution* $\Gamma_\lambda{}^\nu{}_\mu$ *is* (14.5)b. *If this condition is satisfied, then* $\Gamma_\lambda{}^\nu{}_\mu$ *is given by* (11.9).[1]

The proof of these Theorems follows the pattern of the proofs of the corresponding Theorems (11.1), (11.2), (11.3) and (11.4).

NOTE. Many formulae involved in the previous statements reduce considerably for the second class. Thus, for $K > 0$ we have from I (10.6)

$$\overset{12}{P}_\lambda{}^\nu = \alpha^{-2}\,{}^{(2)}k_\lambda{}^\nu, \quad \overset{12}{Q}_{\omega\mu} = \alpha^{-1}k_{\omega\mu},$$

and according to I (10.10)

$$2\alpha\underset{1}{E}_\lambda{}^\nu = \alpha^{-1}\,{}^{(2)}k_\lambda{}^\nu + k_\lambda{}^\nu$$

$$2\alpha\underset{2}{E}_\lambda{}^\nu = \alpha^{-1}\,{}^{(2)}k_\lambda{}^\nu - k_\lambda{}^\nu.$$

Moreover, the coefficients in the first, ..., fourth row of (10.3)b are

$$\frac{1}{g}, \; 1, \quad \frac{1}{2-g}, \; 1$$

and so on.

Third class

15. Auxiliary results. Sections 15–20 of this part of Chapter II will be devoted to the third class. The method developed in sections 9–14 for the first two classes could also be used in the case of the third class.[2] However, efficient as this method has been for the first two classes, many of its merits are lost in the case of the third class.[3] Therefore, we shall devise a more appropriate method. To this effect we first need some auxiliary results.

The third class is characterized by the conditions

$$(15.1) \qquad k = 0, \quad K = 0, \quad g = 1.$$

It admits a nonholonomic frame exhibited in sections I (12) and

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[1] *Cf.* the Notes following Theorems (5.4) and (5.5).

[2] *Cf.* Hlavatý, V., (19).

[3] For the first two classes the system (3.12) of twenty-four equations reduces substantially to the system (11.3) of three equations. This, however, is not the case for the third class.

I (13), where the equations I (13.1) and I (13.2) have also been established. These equations may be condensed as follows:

(15.2)a $\quad k_1{}^2 = k_2{}^4 = k_{12} = -k_{21} = -k^{42} = k^{24} = 1$

(15.2)b $\quad$ the remaining k_{ij}, k^{ij} and $k_j{}^i = 0$,

and

(15.2)c $\quad {}^{(p)}k_1{}^4 = -{}^{(p)}k_{11} = -{}^{(p)}k^{44} = \delta_2^p \quad p > 1$

(15.2)d $\quad$ the remaining ${}^{(p)}k_{ij}$, ${}^{(p)}k^{ij}$, and ${}^{(p)}k_j{}^i = 0$.

We shall use these equations in the proof of the following

LEMMA (15.1). *The equations*

(15.3)a $$K_{i[24]} = 0$$

(15.3)b $$K_{434} = 0$$

hold in every nonholonomic frame involved in Theorem I (13.3).

PROOF. We have in these nonholonomic components by virtue of (15.1) and (15.2)

(15.4) $$\begin{aligned} 2K_{i[24]} &= K_{ijk}k^{jk} = (\nabla_i k_{kj} + \nabla_j k_{ik} + \nabla_k k_{ij})k^{jk} = \\ &= (\nabla_i k_{kj})k^{jk} = -\tfrac{1}{2}\nabla_i k_{kj}k^{kj} \\ &= -\tfrac{1}{2}\partial_i k_{kj}k^{kj} = -2\partial_i K = 0.\,^{1} \end{aligned}$$

This equation proves the assertion about (15.3)a. Moreover, by virtue of (15.2)

$$K_{434} = 2\nabla_4 k_{43} = 2(\partial_4 k_{43} - \{{}_4{}^j{}_4\}k_{j3} - \{{}_3{}^j{}_4\}k_{4j}) = 0.$$

If we perform the transformation I (13.4) of the frame, we obtain according to I (13.7) and I (13.5)a

$$'K_{434} = \overset{-1}{C}_4{}^4 \overset{-1}{C}_3{}^j \overset{-1}{C}_4{}^4 K_{4j4} = \overset{-1}{C}_3{}^3 K_{434} + \overset{-1}{C}_3{}^4 K_{444} = \varepsilon' K_{434}.$$

This equation shows that (15.3)b is independent of the transformations I (13.4) of the frame.

---

[1] We remind the reader that

$$\nabla_i k_{kj} = A_i^\omega A_k^\mu A_j^\lambda \quad \nabla_\omega k_{\mu\lambda} = \partial_i k_{kj} - \{{}_k{}^l{}_i\}k_{lj} - \{{}_j{}^l{}_i\}k_{kl}$$

where $\partial_i = A_i^\lambda \partial_\lambda$, the Christoffel symbols are defined by (13.1), and the nonholomic frame involved in these equations is to be taken from sections I (12) and I (13).

THEOREM (15.1). *The tensor* $2\underset{1}{S}_{\omega\mu\nu}$ *defined by* (8.5)b *for* $m = 1$ *reduces for the third class to*

$$(15.5) \qquad 2\underset{1}{S}_{\omega\mu\nu} = K_{\omega\mu\nu} - 2k_{[\mu}{}^{\alpha}K_{\omega]\alpha\beta}k_{\nu}{}^{\beta}.$$

PROOF. First we have by virtue of (8.5)a for $m = 1$, (8.1)d and (8.4)a

$$(15.6)\text{a} \qquad \underset{1}{U}_{\nu\lambda\mu} = K_{\nu(\lambda}{}^{\alpha}k_{\mu)\alpha},$$

so that (8.5)b yields

$$(15.6)\text{b} \qquad 2\underset{1}{S}_{\omega\mu\nu} = K_{\omega\mu\nu} - 2k_{[\mu}{}^{\alpha}K_{\omega]\alpha\beta}k_{\nu}{}^{\beta} - 2K_{\nu\alpha\beta}k_{[\omega}{}^{\beta}k_{\mu]}{}^{\alpha}.$$

The nonholonomic components of the last tensor on the right hand side are

$$(15.7)\text{a} \qquad -2K_{iqp}k_{[j}{}^{p}k_{k]}{}^{q}.$$

The only components of the tensor (15.7)a which do not vanish identically by virtue of (15.2)a are

$$\pm 2K_{iqp}k_{[1}{}^{p}k_{2]}{}^{q} = \pm 2K_{i[24]}.$$

Hence, we have according to (15.3)a

$$(15.7)\text{b} \qquad K_{iqp}k_{[j}{}^{p}k_{k]}{}^{q} = 0,$$

so that (15.6)b reduces to (15.5).

THEOREM (15.2). *The tensor* $\underset{1}{S}_{\omega\mu\nu}$ *defined by* (15.5) *is a solution of* (3.12).

PROOF. The nonholonomic forms of (3.12)[1] and (15.5) are

$$(15.8) \qquad 2S_{pqr} - K_{pqr} - 4S_{rij}k_{[p}{}^{i}k_{q]}{}^{j} - 4k_{[p}{}^{i}S_{q]ij}k_{r}{}^{j} = 0$$

$$(15.9) \qquad 2\underset{1}{S}_{pqr} = K_{pqr} - 2k_{[q}{}^{i}K_{p]ij}k_{r}{}^{j}.$$

We shall now show that a solution of (15.8) is $S_{pqr} = \underset{1}{S}_{pqr}$. If in the left-hand member of (15.8) one replaces S by $\underset{1}{S}$ and takes $\underset{1}{S}$ from (15.9), one obtains

$$(15.10) \qquad -2k_{[p}{}^{i}\,{}^{(2)}k_{q]}{}^{j}K_{ikj}k_{r}{}^{k} + 2\,{}^{(2)}k_{[p}{}^{i}K_{q]ij}\,{}^{(2)}k_{r}{}^{j} + 2k_{[p}{}^{i}k_{q]}{}^{j}K_{ijk}\,{}^{(2)}k_{r}{}^{k}.$$

---

[1] See (9.8).

Consider the first of the three tensors in (15.10). The only components which do not vanish identically by means of (15.2) are those for $p = 2,\ 1,\ q = 1,\ 2,\ r = 1,\ 2,$ *i.e.*,

$$-\,2k_{[2}{}^{i\,(2)}k_{1]}{}^{j}(K_{i2j}\delta^{1}_{r} + K_{i4j}\delta^{2}_{r}) = -\,K_{424}\delta^{1}_{r} = -\,2K_{4[24]}\delta^{1}_{r}.$$

Hence, this component vanishes by virtue of (15.3)a. Similar reasoning based on (15.3)a,b shows that each of the remaining two tensors in (15.10) is equal to zero. Hence, the tensor (15.10) is equal to zero.

16. The tensor Y. Using the previous result we shall be able to find the tensor Y defined in Theorem (5.1). For this purpose we shall need the following tensors:

$$(16.1)\mathrm{a}\qquad \underset{1}{X}{}^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\mathrm{def}}{=} X^{\alpha\beta\gamma}_{\omega\mu\nu} - \delta^{[\alpha\beta]\gamma}_{[\omega\mu]\nu} =$$

$$= 2\delta^{[\alpha}_{\nu}k_{[\mu}{}^{\beta]}k_{\omega]}{}^{\gamma} + 2\delta^{[\alpha}_{[\omega}k_{\mu]}{}^{\beta]}k_{\nu}{}^{\gamma}$$

$$(16.1)\mathrm{b}\qquad {}''Y^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\mathrm{def}}{=} \delta^{[\alpha\beta]\gamma}_{[\omega\mu]\nu} - 2\delta^{[\alpha}_{[\omega}k_{\mu]}{}^{\beta]}k_{\nu}{}^{\gamma}$$

$$(16.1)\mathrm{c}\qquad T^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\mathrm{def}}{=} -\,2\delta^{[\alpha}_{\nu}k_{[\omega}{}^{\beta]}k_{\mu]}{}^{\gamma} + 2k_{\nu}{}^{[\alpha}(k_{[\omega}{}^{\beta]\,(2)}k_{\mu]}{}^{\gamma} - {}^{(2)}k_{[\mu}{}^{\beta]}k_{\omega]}{}^{\gamma}) +$$

$$+\,2\,{}^{(2)}k_{\nu}{}^{\gamma}(\delta^{[\alpha}_{[\mu}\,{}^{(2)}k_{\omega]}{}^{\beta]} - k_{[\mu}{}^{[\beta}k_{\omega]}{}^{\alpha]})$$

$$(16.1)\mathrm{d}\qquad {}'Y^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\mathrm{def}}{=} {}''Y^{\alpha\beta\gamma}_{\omega\mu\nu} - T^{\alpha\beta\gamma}_{\omega\mu\nu}.$$

Moreover, we shall symbolize any equation of the type (5.4) by

$$(16.2)\qquad YX = \delta.$$

The tensor $''Y$ enables us to write (15.5) in a more concise form

$$(16.3)\mathrm{a}\qquad 2\underset{1}{S}_{\omega\mu\nu} = {}''Y^{\alpha\beta\gamma}_{\omega\mu\nu}K_{\alpha\beta\gamma}.$$

First we will prove the following

LEMMA (16.1). *We have*

$$(16.3)\mathrm{b}\qquad {}''YX = \delta + T$$

$$(16.3)\mathrm{c}\qquad (T\underset{1}{X})\underset{1}{X} = 0,$$

where T is the tensor defined by (16.1)c.

PROOF. Using (15.2)c,d we see that

$$(16.4)\qquad {}^{(2)}k_{[\omega}{}^{\alpha}\,{}^{(2)}k_{\mu]}{}^{\beta} = 0.$$

The equations (16.3)b,c follow by a simple inspection which is based on (15.2)a–d, and (16.4).

THEOREM (16.1). *Put*

$$(16.5)\qquad Y^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\text{def}}{=} {}'Y^{\alpha\beta\gamma}_{\omega\mu\nu} + T^{\xi\eta\zeta}_{\omega\mu\nu}\underset{1}{X}{}^{\alpha\beta\gamma}_{\xi\eta\zeta}$$

$$= {}''Y^{\alpha\beta\gamma}_{\omega\mu\nu} - T^{\alpha\beta\gamma}_{\omega\mu\nu} + T^{\xi\eta\zeta}_{\omega\mu\nu}\underset{1}{X}{}^{\alpha\beta\gamma}_{\xi\eta\zeta}.$$

Then Y is the tensor defined in Theorem (5.1).

PROOF. Using (16.1)d we obtain by virtue of (16.3)b and (16.1)a the following relationship:

$$'YX = {}''YX - TX = \delta + T - T - T\underset{1}{X}$$

$$= \delta - T\underset{1}{X}.$$

Therefore, by virtue of (16.5) and (16.3)c

$$YX = {}'YX + (T\underset{1}{X})X = \delta - T\underset{1}{X} + T\underset{1}{X} + (T\underset{1}{X})\underset{1}{X} =$$

$$= \delta + (T\underset{1}{X})\underset{1}{X} = \delta.$$

This equation holds if and only if the matrix $((X))$ is of rank $m = 24$. Then, of course, there is only one tensor Y satisfying it. Therefore, the tensor Y defined in Theorem (5.2) must be the same as the one defined by (16.5)a.

17. The tensor $S_{\omega\mu\nu}$. According to Theorem (15.2) the tensor $\underset{1}{S}_{\omega\mu\nu}$ is *a* solution of (3.12). In this section we shall prove that it is the *only* solution of (3.12). For this purpose we shall need the following

LEMMA (17.1). *We have*

$$(17.1)\text{a}\qquad K_{\alpha\beta\gamma}T^{\alpha\beta\gamma}_{\omega\mu\nu} = 0$$

$$(17.1)\text{b}\qquad K_{\alpha\beta\gamma}T^{\xi\eta\zeta}_{\omega\mu\nu}\underset{1}{X}{}^{\alpha\beta\gamma}_{\xi\eta\zeta} = 0.$$

PROOF. According to (16.1)c the nonholonomic components of the tensor on the left-hand side of (17.1)a are

$$(17.2)\qquad K_{pqr}[-2\delta^{[p}_{k}k_{[i}{}^{q]}k_{j]}{}^{r} + 2k_{k}{}^{[p}(k_{[i}{}^{q]}\,{}^{(2)}k_{j]}{}^{r} - {}^{(2)}k_{[j}{}^{q]}k_{i]}{}^{r})$$

$$+ 2\,{}^{(2)}k_{k}{}^{r}(\delta^{[p}_{[j}\,{}^{(2)}k_{i]}{}^{q]} - k_{[j}{}^{[q}k_{i]}{}^{p]})].$$

Consider the first of the three tensors in (17.2). The only components of this tensor which are not identically equal to zero by virtue of (15.2) are those for which $ij = 12$ or $ij = 21$. Since the tensor under consideration is skew symmetric in ij, it is sufficient to investigate the first possibility

$$-2\delta^{[p}_{k}k_{[1}{}^{q]}k_{2]}{}^{r}K_{pqr} = -k_{[1}{}^{q}k_{2]}{}^{r}K_{kqr} + k_{[1}{}^{p}k_{2]}{}^{r}K_{pkr} =$$
$$= -K_{k[24]} - K_{k[24]} = -2K_{k[24]}.$$

Hence, the tensor under consideration is equal to zero by virtue of (15.3)a. The vanishing of the remaining two tensors in (17.2) may be proved in a similar manner by means of (15.3)a,b. The same method, based on (15.2) and (15.3), leads to (17.1)b.

The following two Theorems will be proved simultaneously.

THEOREM (17.1)a. *The system* (3.12) *always admits a unique solution* $S_{\omega\mu\nu}$.

THEOREM (17.1)b. *The unique solution of* (3.12) *is*

$$(17.3) \qquad 2S_{\omega\mu\nu} = K_{\omega\mu\nu} - 2k_{[\mu}{}^{\alpha}K_{\omega]\alpha\beta}k_{\nu}{}^{\beta} =$$
$$= K_{\alpha\beta\gamma}\ {}''Y^{\alpha\beta\gamma}_{\omega\mu\nu}$$
$$= K_{\alpha\beta\gamma}\ Y^{\alpha\beta\gamma}_{\omega\mu\nu}$$

where the tensors $''Y$ *and* Y *are defined by* (16.1)b *and* (16.5) *respectively.*

PROOF. Remembering (17.1) we obtain from (16.5)

$$(17.4) \qquad K_{\alpha\beta\gamma}Y^{\alpha\beta\gamma}_{\omega\mu\nu} = K_{\alpha\beta\gamma}({}''Y^{\alpha\beta\gamma}_{\omega\mu\nu} - T^{\alpha\beta\gamma}_{\omega\mu\nu} + T^{\xi\eta\zeta}_{\omega\mu\nu}\underset{1}{X}{}^{\alpha\beta\gamma}_{\xi\eta\zeta})$$
$$= K_{\alpha\beta\gamma}{}''Y^{\alpha\beta\gamma}_{\omega\mu\nu}\ .$$

Since the tensor Y is the unique one satisfying (5.4), we obtain from (3.12) the only solution (5.10), and this solution satisfies (17.3) by virtue of (16.3)a and (17.4). The construction of the unique tensor Y (and, by the same token, of the unique solution (17.3)) did not require that any particular conditions be imposed on the tensor $k_{\lambda\mu}$ of the third class.

NOTE. Compare Theorems (11.1), (14.3) and the last of the equations (15.1).

18. The tensor $U^{\nu}{}_{\lambda\mu}$ and the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. In order to find a suitable form for the tensor $U^{\nu}{}_{\lambda\mu}$ defined by (3.2), the following Lemma will be necessary.

LEMMA (18.1). *Put*

$$Q_{\nu\mu\lambda} \overset{\text{def}}{=} 2k_{\nu}{}^{\beta}k_{(\mu}{}^{\alpha}k_{\lambda)}{}^{\gamma}K_{\alpha\beta\gamma}. \tag{18.1)a}$$

Then

$$Q_{\nu\mu\lambda} = 0. \tag{18.1)b}$$

PROOF. Among the nonholonomic components Q_{rpq} of $Q_{\nu\mu\lambda}$ the only ones which do not identically vanish by virtue of (15.2) are those for $r, p, q = 1, 2$. We have

$$Q_{r11} = 2k_{r}{}^{j}k_{(1}{}^{i}k_{1)}{}^{k}K_{ijk} = 4\delta_{r}^{2}K_{2[42]},$$

and similarly,

$$Q_{r12} = Q_{r21} = 2(\delta_{r}^{1}K_{2[24]} - \delta_{r}^{2}K_{4[24]})$$

$$Q_{r22} = 4\delta_{r}^{1}K_{4[24]}.$$

Hence, these components vanish by virtue of (15.3)a so that (18.1)b holds.

In the next Theorem we shall use the tensor $H_{\nu\lambda\mu}$ defined by (8.1)d so that

$$k_{\mu}{}^{\alpha}K_{\lambda\nu\alpha} = -2H_{\nu\lambda\mu} - k_{\lambda}{}^{\alpha}K_{\mu\nu\alpha}. \tag{18.2}$$

THEOREM (18.1). *The tensor $U_{\nu\lambda\mu}$ is given by*

$$\begin{aligned} U_{\nu\lambda\mu} &= K_{\nu(\lambda}{}^{\alpha}k_{\mu)\alpha} + k_{\gamma}{}^{\beta}[k_{(\mu}{}^{\gamma}K_{\lambda)\alpha\beta}k_{\nu}{}^{\alpha} - K_{\nu\alpha\beta}k_{(\lambda}{}^{\alpha}k_{\mu)}{}^{\gamma}] \\ &= H_{\gamma\alpha\beta}[\delta_{\nu(\lambda\mu)}^{\gamma\alpha\beta} + 2\delta_{\nu}^{\alpha}k_{(\lambda}{}^{\gamma}k_{\mu)}{}^{\beta} - 2\delta_{(\lambda}^{\beta}k_{\mu)}{}^{\alpha}k_{\nu}{}^{\gamma}]. \end{aligned} \tag{18.3}$$

PROOF. The first equation results from substituting from (17.3) into (3.2). On the other hand, we have by virtue of (18.2)

$$\begin{aligned} &-k_{\gamma}{}^{\beta}K_{\nu\alpha\beta}k_{(\lambda}{}^{\alpha}k_{\mu)}{}^{\gamma} = \\ &= (2H_{\alpha\nu\gamma} + k_{\nu}{}^{\beta}K_{\gamma\alpha\beta})k_{(\lambda}{}^{\alpha}k_{\mu)}{}^{\gamma} \\ &= 2H_{\alpha\nu\gamma}k_{(\lambda}{}^{\alpha}k_{\mu)}{}^{\gamma}. \end{aligned} \tag{18.4)a}$$

Moreover, one obtains according to (18.1)b and (18.2)

$$\begin{aligned}(18.4)\text{b}\qquad & k_{(\mu}{}^{\gamma}k_{|\gamma|}{}^{\beta}K_{\lambda)\alpha\beta}k_{\nu}{}^{\alpha} = \\ &= -k_{(\mu}{}^{\gamma}(2H_{|\alpha\gamma|\lambda)} + k_{\lambda)}{}^{\delta}K_{\gamma\alpha\delta})k_{\nu}{}^{\alpha} = \\ &= -2H_{\gamma\alpha(\lambda}k_{\mu)}{}^{\alpha}k_{\nu}{}^{\gamma} - \tfrac{1}{2}Q_{\nu\mu\lambda} = \\ &= -2H_{\gamma\alpha(\lambda}k_{\mu)}{}^{\alpha}k_{\nu}{}^{\gamma}.\end{aligned}$$

Upon substituting from (18.4)a,b into the second row of (18.3), and using (8.1)d, one obtains the third row.

NOTE. The equations (7.2) and (18.3) are useful in the proof of Corollary (7.1): The necessity follows directly from (7.2), while the sufficiency is a direct consequence of (18.3).

THEOREM (18.2). *The system* (2.1) *always admits a unique solution, namely,*

$$\begin{aligned}(18.5)\qquad \Gamma_{\lambda}{}^{\nu}{}_{\mu} &= \{\lambda{}^{\nu}{}_{\mu}\} + \tfrac{1}{2}K_{\lambda\mu}{}^{\nu} - k_{[\mu}{}^{\alpha}K_{\lambda]\alpha\beta}k^{\nu\beta} \\ &+ h^{\omega\nu}K_{\gamma(\alpha}{}^{\delta}k_{\beta)\delta}[\delta^{\gamma}_{\omega}{}^{\alpha\beta}_{(\lambda\mu)} + 2\delta^{\alpha}_{\omega}k_{(\lambda}{}^{\gamma}k_{\mu)}{}^{\beta} - 2\delta^{\beta}_{(\lambda}k_{\mu)}{}^{\alpha}k_{\omega}{}^{\gamma}].\end{aligned}$$

The proof, based on Theorems (3.1), (17.1) and (18.1), is quite obvious.

NOTE I. *Cf.* Theorems (11.4) and (14.5).

NOTE II. In the first of his three famous papers on the affine field law, Schrödinger, E., (1) p. 166 writes: "In the general case [1] it is next to impossible to produce it [2] in a surveyable tensorial form. [3] (This is hard to believe unless one has tried)". We believe that (11.9) as well as (13.3) and (18.5) satisfy this requirement.

19. Auxiliary results. In this section we shall derive some results useful for the next section. In particular, we shall consider the nonholonomic components $\{_{j}{}^{i}{}_{k}\}$ defined by (13.1)a, where, of course, the vectors $\underset{j}{a}{}^{\nu}$, $\overset{i}{a}_{\lambda}$ are to be taken from sections I (12) and I (13).

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[1] *i.e.*, for $g_{\lambda\mu} \neq \pm g_{\mu\lambda}$.

[2] *i.e.*, the solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ of (2.1).

[3] *i.e.*, $\Gamma_{\lambda}{}^{\nu}{}_{\mu} - \{\lambda{}^{\nu}{}_{\mu}\}$ (or $\Gamma_{x}{}^{t}{}_{y} - \{x{}^{t}{}_{y}\}$) is expressed in a surveyable tensorial form. For the case where this tensor is not expressed in a "surveyable" tensorial form *cf.* Einstein, A. — Kaufmann, B., (2).

LEMMA (19.1). *The following equation holds independently of the frame transformations* I (13.4):

$$\{{}_4{}^1{}_i\} = 0. \tag{19.1}$$

PROOF. We have according to (13.1)a and I (12.12)

$$\{{}_4{}^1{}_i\} = - \underset{4}{a}^{\lambda} \underset{i}{a}^{\mu} \nabla_{\mu} \overset{1}{a}_{\lambda} = \overset{1}{a}^{\lambda} \underset{i}{a}^{\mu} \nabla_{\mu} \overset{1}{a}_{\lambda} =$$

$$= \tfrac{1}{2} \underset{i}{a}^{\mu} (\nabla_{\mu} \overset{1}{a}^{\lambda} \overset{1}{a}_{\lambda}) = 0.$$

LEMMA (19.2). *Any one of the conditions*

$$K_{234} = 0, \tag{19.2)a}$$

$$K_{342} = 0, \tag{19.2)b}$$

$$K_{423} = 0 \tag{19.2)c}$$

is equivalent to

$$\{{}_3{}^1{}_4\} = 0. \tag{19.2)d}$$

The equations (19.2) *are independent of the frame transformations* I (13.4).

PROOF. We have by virtue of (3.5), (19.1) and (15.2)a

$$\begin{aligned} K_{234} &= \nabla_2 k_{43} + \nabla_3 k_{24} + \nabla_4 k_{23} = \nabla_3 k_{24} + \nabla_4 k_{23} = \\ &= - [\{{}_4{}^1{}_3\} k_{21} + \{{}_3{}^1{}_4\} k_{21}] = \{{}_3{}^1{}_4\}, \end{aligned} \tag{19.3)a}$$

and similarly,

$$- K_{423} = \{{}_3{}^1{}_4\} = - K_{342}. \tag{19.3)b}$$

Hence, each of the equations (19.2)a,b,c is equivalent to (19.2)d. Moreover, we have according to I (12.12), I (13.4) and I (13.5)

$${}'\{{}_3{}^1{}_4\} = - {}'\underset{3}{a}^{\lambda}\, {}'\underset{4}{a}^{\mu} \nabla_{\mu}\, {}'\overset{1}{a}_{\lambda} = {}'\underset{3}{a}^{\lambda}\, {}'\underset{4}{a}^{\mu} \nabla_{\mu}\, {}'\underset{4}{a}_{\lambda} =$$

$$\varepsilon' \underset{3}{a}^{\lambda} \underset{4}{a}^{\mu} \nabla_{\mu} \underset{4}{a}_{\lambda} + \tfrac{1}{2} \varepsilon \varepsilon' u \sin v \underset{4}{a}^{\mu} \nabla_{\mu} \underset{4}{a}_{\lambda} \underset{4}{a}^{\lambda} =$$

$$= - \varepsilon' \underset{3}{a}^{\lambda} \underset{4}{a}^{\mu} \nabla_{\mu} \overset{1}{a}_{\lambda} = \varepsilon' \{{}_3{}^1{}_4\},$$

and this equation yields the desired invariance.

LEMMA (19.3). *The conditions*

(19.4)a $$K_{144} = 0,$$

(19.4)b $$K_{2[41]} = 0,$$

(19.4)c $$K_{322} = 0$$

are in turn equivalent to

(19.5)a $$\{{}_4{}^2{}_4\} = 0,$$

(19.5)b $$\{{}_4{}^2{}_2\} = 0,$$

(19.5)c $$\{{}_3{}^1{}_2\} = 0.$$

The equations (19.4) *and* (19.5) *are independent of the frame transformations* I (13.4).

The proof follows the pattern of the proof of Lemma (19.2).

LEMMA (19.4). *The conditions* (19.2) *and* (19.4) *are equivalent to the system*

(19.6) $$\underset{u+2}{a^\lambda}\ \underset{v}{a^\mu} \nabla_\mu \overset{u}{a}_\lambda = 0$$

$$(u = 1, 2; \quad v = 2, 4)$$

which does not depend on the frame transformations I (13.4).

The proof follows from previous Lemmas and from (13.1)a.

NOTE. According to Lemma (19.4) the conditions (19.6) have an intrinsic meaning. The question arises as to whether they could be satisfied.

THEOREM (19.1). *The set of all tensors* $k_{\lambda\mu}$ *of the third class for which the conditions* (19.6) $\left\{\begin{matrix}\textit{are}\\ \textit{are not}\end{matrix}\right\}$ *satisfied is not an empty set.*

PROOF. Let us consider a tensor $g_{\lambda\mu}$ given by the following matrix

(19.7) $$((g_{\lambda\mu})) = \left(\left(\begin{matrix} 0 & \gamma^{-1} & 0 & -1 \\ -\gamma^{-1} & \gamma^{-2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}\right)\right)$$

where $\gamma \neq 0$ is a real function of coordinates. The corresponding tensor $g_{[\lambda\mu]} = k_{\lambda\mu}$ is obviously of the third class. A simple inspection shows that the nonholonomic frame for which I (13.3) holds is constituted by the vectors

$$(19.8)\qquad \begin{array}{llll} \underset{1}{a}^{\nu}(1 & 0 & 0 & 0) \\ \underset{2}{a}^{\nu}(0 & \gamma & 0 & 0) \\ \underset{3}{a}^{\nu}(0 & 0 & 1 & 0) \\ \underset{4}{a}^{\nu}(0 & 0 & 0 & 1) \end{array} \qquad \begin{array}{llll} \overset{1}{a}_{\lambda}(1 & 0 & 0 & 0) \\ \overset{2}{a}_{\lambda}(0 & \gamma^{-1} & 0 & 0) \\ \overset{3}{a}_{\lambda}(0 & 0 & 1 & 0) \\ \overset{4}{a}_{\lambda}(0 & 0 & 0 & 1). \end{array}$$

Hence, we have

$$(19.9)\text{a}\qquad \underset{4}{a}^{\lambda}\underset{2}{a}^{\mu}\nabla_{\mu}\overset{2}{a}_{\lambda} = -\{\lambda{}^{\nu}{}_{\mu}\}\underset{4}{a}^{\lambda}\underset{2}{a}^{\mu}\overset{2}{a}_{\nu} = \\ = -\{{}_{\text{IV}}{}^{\text{II}}{}_{\text{II}}\} = \partial_{\text{IV}} \ln \gamma,$$

and in a similar manner,

$$(19.9)\text{b}\qquad \underset{4}{a}^{\lambda}\underset{4}{a}^{\mu}\nabla_{\mu}\overset{2}{a}_{\lambda} = \underset{3}{a}^{\lambda}\underset{4}{a}^{\mu}\nabla_{\mu}\overset{1}{a}_{\lambda} = \underset{3}{a}^{\lambda}\underset{2}{a}^{\mu}\nabla_{\mu}\overset{1}{a}_{\lambda} = 0.$$

Hence, if $\gamma \begin{Bmatrix}\text{is} \\ \text{is not}\end{Bmatrix}$ a function of x^{IV}, the conditions (19.6) $\begin{Bmatrix}\text{are not} \\ \text{are}\end{Bmatrix}$ satisfied.

The previous results will be helpful in the next section.

20. Equations (11.1) and (17.3). In order to find the solution of (3.12) (necessary for the construction of the solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ of (2.1)) we developed for the third class a method different from the one used for the first two classes. In this way we obtained the solution in the form given by (17.3), while the previous method yielded for the first two classes the solution in the form (11.1). The question arises as to whether this latter equation cannot also be used in the case of the third class. This question is very important for later purposes. If the answer is affirmative, we can deal with all three classes simultaneously as we did with the first two classes. In the case of a negative answer we have to deal with the third class separately.

In order to find the answer to this important question we have first to clarify the role played by the tensor $L_{\omega\mu\nu}$ in the case of the third class.

THEOREM (20.1). *If we define the tensor* $L_{\omega\mu\nu}$ *by* (10.6),[1] *then*

(20.1)a $$L_{\omega\mu\nu} = K_{\omega\mu\nu}.$$

PROOF. By Theorem I (7.5)a, the basic scalars $\underset{1}{\lambda}, \dots, \underset{4}{\lambda}$ are all equal to zero for the third class. Hence, we have according to (9.2)

(20.2) $$\underset{ijk}{\lambda} = 1,$$

so that (10.6)b reduces to

(20.1)b $$L_{xyz} = K_{xyz}$$

equivalent to (20.1)a.

After having introduced the tensor $L_{\omega\mu\nu}$ we are able to prove the following

THEOREM (20.2). *In order that* (11.1) *represent the solution of* (3.12) *for the third class also, it is necessary and sufficient that the equations* (19.6) *hold.*

PROOF. A necessary and sufficient condition follows at once from (11.1), (20.1)a and (17.3) in the form

(20.2)a $$-\,2k_{[\mu}{}^{\alpha}K_{\omega]\alpha\beta}k_{\nu}{}^{\beta} = 3K_{[\alpha\beta\nu]}k_{\omega}{}^{\alpha}k_{\mu}{}^{\beta}.$$

However, we shall use its nonholonomic form

(20.2)b $$k_k{}^p(k_i{}^qK_{jqp} - k_j{}^qK_{iqp}) = 3K_{[pqk]}k_i{}^pk_j{}^q.$$

This system splits by virtue of (15.2) into three different sub-systems:[2]

I. Put $k = c$, $i = a$, $j = f$. Then (20.2)b reduces to

(20.3) $$K_{fa^*c^*} = 0.$$

---

[1] The nonholonomic components K_{xyz} are constructed here with respect to the nonholonomic frame of the third class.

[2] In the present discussion we denote by a, b, c (e, f, g) the indices which run from 1 to 2 (from 3 to 4). Moreover, we shall use the "starred" indices a^*, b^*, c^* which are defined as follows:

If $a = 1, 2,$

then $a^* = 2, 4,$

and similarly for b^*, c^*.

Using (15.3) we see that this equation is identically satisfied for several combinations of the indices. It is not an identity for the following cases. For $f = 3$, $a = c = 1$

$$K_{322} = 0, \tag{20.4)a}$$

for $f = 3$, $a = 1$, $c = 2$

$$K_{324} = 0, \tag{20.4)b}$$

and for $f = 3$, $a = 2$, $c = 1$

$$K_{342} = 0. \tag{20.5}$$

According to Lemma (19.2) the equation (20.5) is a consequence of (20.4)b so we may disregard it. It is clear that $k = c$, $i = f$, $j = a$ leads to the same result.

II. Put $k = c$, $i = 1$, $j = 2$. Then (20.2)b reduces to

$$- K_{14c*} = 3K_{[24c]}. \tag{20.6}$$

This condition splits in two: for $c = 2$

$$K_{144} = 0 \tag{20.4)c}$$

and for $c = 1$

$$K_{2[14]} = 0. \tag{20.4)d}$$

It is clear that $k = c$, $i = 2$, $j = 1$ would yield the same result. Thus, all possibilities for $k = c$ are exhausted.

III. For $k = f$ the system (20.2)b reduces to

$$K_{[pqf]} k_i{}^p k_j{}^q = 0. \tag{20.7}$$

The condition is not an identity if $i = 1$, $j = 2$

$$K_{[24f]} = 0,$$

so that we obtain for $f = 3$ the condition

$$K_{[243]} = 0 \tag{20.8}$$

which by virtue of Lemma (19.2) and condition (20.4)b is automatically satisfied. It is clear that $i = 2$, $j = 1$ yields the same result. This exhausts all possibilities for $k = f$. Hence, the system (20.2)b splits into (20.4)a–d, *i.e.*, into (19.4) and (19.2)a. According to Lemma (19.4) this system is equivalent to (19.6).

THEOREM (20.3). *As far as the problem B is concerned*[1] *there are two unified field theories. One for the first two classes based on* (11.9), *another for the third class based on* (18.5).[2] *If and only if the conditions* (19.6) *are satisfied, both theories coincide in the sense that* (11.9) *holds for all three classes.*[3]

The proof based on (3.1) and on Theorem (20.2) is obvious.

21. Special form of $g_{\lambda\mu}$.

The system (2.1) has been solved by several authors for *special forms of* $g_{\lambda\mu}$.[4] In this section we shall investigate the restrictions which lead to such special forms in the case of the first two classes. To this effect we first need the following

LEMMA (21.1). *Let* $\overset{x}{a}_\lambda$ *be basic vectors. If and only if*

$$(21.1)\qquad \underset{x}{b}^\nu = \underset{x}{\rho}\,\underset{x}{a}^\nu,\quad \overset{x}{b}_\lambda = \overset{x}{\rho}\,\overset{x}{a}_\lambda,\quad \overset{x}{\rho}\,\underset{x}{\rho} = 1,$$

the nonholonomic components g_{ij} *with respect to the nonholonomic frame* $\underset{x}{b}^\nu$, $\overset{x}{b}_\lambda$ *are*

$$(21.2)\qquad ((g_{ij})) = \left(\left(\begin{matrix} 0 & h_{12}+k_{12} & 0 & 0 \\ h_{21}+k_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{34}+k_{34} \\ 0 & 0 & h_{43}+k_{43} & 0 \end{matrix}\right)\right).$$

$$h_{12}h_{34} \neq 0.$$

PROOF. Assume first that (21.2) are satisfied for a nonholonomic frame $\underset{x}{b}^\nu$, $\overset{x}{b}_\lambda$ where, of course,

$$h_{xy} \overset{\text{def}}{=} h_{\lambda\mu}\underset{x}{b}^\lambda \underset{y}{b}^\mu, \qquad k_{xy} \overset{\text{def}}{=} k_{\lambda\mu}\underset{x}{b}^\lambda \underset{y}{b}^\mu.$$

Then we see from (21.2) that $\underset{x}{b}^\nu$ are null vectors and that the bivectors $\underset{1}{b}{}^{[\nu}\underset{2}{b}{}^{\lambda]}$ and $\underset{3}{b}{}^{[\nu}\underset{4}{b}{}^{\lambda]}$ are totally perpendicular. The vectors $\underset{x}{b}{}^{\nu}$ and $\overset{x}{b}_{\lambda}$ constitute a nonholonomic frame so that

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[1] *i.e.*, the solution $\Gamma_\lambda{}^\nu{}_\mu$ of (2.1) expressed explicitly in terms of $g_{\lambda\mu}$.

[2] Of course, both formulae may be expressed *formally* in the same way by means of (5.9).

[3] The question under what conditions (18.5) holds for all three classes is left open.

[4] See footnote 3 on page 50.

(21.3)a $$\underset{j}{b}{}^{\nu}\overset{i}{b}_{\nu} = \delta^{i}_{j},$$

and the nonholonomic components $k_j{}^i$ of $k_\lambda{}^\nu$ are defined by

(21.3)b $$k_j{}^i = k_\lambda{}^\nu \underset{j}{b}{}^{\lambda}\overset{i}{b}_{\nu}.$$

Then we have from (21.2)

(21.3)c $$k_1{}^z = k_\lambda{}^\nu \underset{1}{b}{}^{\lambda}\overset{z}{b}_{\nu} = 0, \qquad z = 3, 4$$

so that $k_\lambda{}^\nu \underset{1}{b}{}^{\lambda}$ and $\overset{z}{b}{}^{\nu}$ are perpendicular

(21.4)a $$k_\lambda{}^\nu \underset{1}{b}{}^{\lambda} = \underset{1}{\alpha}\,\underset{1}{b}{}^{\nu} + \beta \underset{2}{b}{}^{\nu}.$$

Multiplying this equation by $\overset{1}{b}_{\nu}$, $\overset{2}{b}_{\nu}$, we find according to (21.3) and (21.2)

$$k_1{}^1 = \underset{1}{\alpha}, \qquad k_1{}^2 = \beta = 0$$

so that (21.4)a reduces to

(21.4)b $$k_\lambda{}^\nu \underset{1}{b}{}^{\lambda} = \underset{1}{\alpha}\,\underset{1}{b}{}^{\nu}.$$

Applying the same method to $k_j{}^i$ we obtain

(21.5) $$k_\lambda{}^\nu \underset{x}{b}{}^{\lambda} = \underset{x}{\alpha}\,\underset{x}{b}{}^{\nu}.$$

Hence, the null vectors $\underset{x}{b}{}^{\nu}$ satisfy the first set of (21.1), and by (21.3)a the vectors $\overset{x}{b}_{\lambda}$ satisfy the second set (21.1). The third set is also a consequence of (21.3)a.

The sufficiency of (21.1) for (21.2) is obvious.

In the next Lemma we shall use the tensors $\underset{x}{E}_\lambda{}^\nu$ defined by I (10.10).

LEMMA (21.2). *The tensor* $\underset{x}{E}_{[\omega}{}^{\alpha}\partial_{\mu}\underset{x}{E}_{\lambda]}{}^{\beta}$ *satisfies the condition*

$$\underset{x}{E}_{[\omega}{}^{\alpha}\partial_{\mu}\underset{x}{E}_{\lambda]}{}^{\beta} = \underset{x}{a}^{\alpha}\underset{x}{a}^{\beta}\overset{x}{a}_{[\omega}\partial_{\mu}\overset{x}{a}_{\lambda]}. \tag{21.6}$$

PROOF. We have by virtue of I (10.11)

$$\begin{aligned}\underset{x}{E}_{[\omega}{}^{\alpha}\partial_{\mu}\underset{x}{E}_{\lambda]}{}^{\beta} &= \underset{x}{a}^{\alpha}\overset{x}{a}_{[\omega}[(\partial_{\mu}\overset{x}{a}_{\lambda]})\underset{x}{a}^{\beta} + \overset{x}{a}_{\lambda}\partial_{\mu]}\underset{x}{a}^{\beta}] \\ &= \underset{x}{a}^{\alpha}\underset{x}{a}^{\beta}\overset{x}{a}_{[\omega}\partial_{\mu}\overset{x}{a}_{\lambda]}.\end{aligned}$$

THEOREM (21.1). *In order that there be a coordinate system in which* $g_{\lambda\mu}$ *acquires the form*

$$((g_{\lambda\mu})) = \left(\begin{pmatrix} 0 & h_{\text{I II}}+k_{\text{I II}} & 0 & 0 \\ h_{\text{II I}}+k_{\text{II I}} & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{\text{III IV}}+k_{\text{III IV}} \\ 0 & 0 & h_{\text{IV III}}+k_{\text{IV III}} & 0 \end{pmatrix}\right) \tag{21.7}$$

it is necessary that

$$\underset{x}{E}_{[\omega}{}^{\alpha}\partial_{\mu}\underset{x}{E}_{\lambda]}{}^{\beta} = 0. \tag{21.8a}$$

This set of 4 × 10 × 4 *conditions is equivalent to the* 16 *conditions*

$$\overset{x}{a}_{[\omega}\partial_{\mu}\overset{x}{a}_{\lambda]} = 0. \tag{21.8b}$$

PROOF. According to Lemma (21.1) a necessary and sufficient condition for the existence of a coordinate system in which $g_{\lambda\mu}$ acquires the form (21.7) is that the four vectors $\overset{x}{b}_{\lambda}$ be gradients. By virtue of (21.1) this condition is equivalent to

$$\partial_{[\mu}\overset{x}{\rho}\,\overset{x}{a}_{\lambda]} = 0.$$

If this equation is satisfied, then (21.8)b holds, and the remaining part of the proof follows from Lemma (21.2).

NOTE I. If we substitute into (21.8)a from I (10.10) and I (10.6), we obtain a set of conditions for $g_{\lambda\mu}$ where *the basic vectors are not explicitly involved.*

NOTE II. The special forms of $g_{\lambda\mu}$ used by different authors for solving (2.1) are of three types

$$\left(\!\!\left(\begin{matrix} h_{\mathrm{I\,I}} & 0 & 0 & k_{\mathrm{I\,IV}} \\ 0 & h_{\mathrm{II\,II}} & k_{\mathrm{II\,III}} & 0 \\ 0 & k_{\mathrm{III\,II}} & h_{\mathrm{III\,III}} & 0 \\ k_{\mathrm{IV\,I}} & 0 & 0 & h_{\mathrm{IV\,IV}} \end{matrix}\right)\!\!\right), \quad \left(\!\!\left(\begin{matrix} h_{\mathrm{I\,I}} & 0 & k_{\mathrm{I\,III}} & 0 \\ 0 & h_{\mathrm{II\,II}} & 0 & k_{\mathrm{II\,IV}} \\ k_{\mathrm{III\,I}} & 0 & h_{\mathrm{III\,III}} & 0 \\ 0 & k_{\mathrm{IV\,II}} & 0 & h_{\mathrm{IV\,IV}} \end{matrix}\right)\!\!\right), \tag{21.9}$$

$$\left(\!\!\left(\begin{matrix} h_{\mathrm{I\,I}} & k_{\mathrm{I\,II}} & 0 & 0 \\ k_{\mathrm{II\,I}} & h_{\mathrm{II\,II}} & 0 & 0 \\ 0 & 0 & h_{\mathrm{III\,III}} & k_{\mathrm{III\,IV}} \\ 0 & 0 & k_{\mathrm{IV\,III}} & h_{\mathrm{IV\,IV}} \end{matrix}\right)\!\!\right).$$

All the h's are different from zero. A simple inspection shows that each of these three types could be reduced to (21.7) *by linear transformations of coordinates.* Therefore, *Theorem* (21.1) *also holds for these special types.* In other words, if we are dealing with one of the forms (21.9), *we have to impose on* $g_{\lambda\mu}$ *a set of new conditions* (21.8)a. This is a very substantial restriction of generality which could considerably reduce the possibility of physical application [1].

This concludes the investigation of the solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ of (2.1) in the general case. The next chapter is devoted to the singular cases, where the conditions (9.4)b and (14.5)b are not satisfied.

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[1] See section 3 in the fourth chapter.

CHAPTER III

SPECIAL PROBLEMS AND SINGULAR CASES

PART I

The tensor $N_{\lambda\omega}{}^{\nu}$ [1]

1. Prerequisites. In this chapter we shall consider the solution $S_{\alpha\beta\gamma}$ of

(1.1)a [2] $$2S_{\alpha\beta\gamma}X^{\alpha\beta\gamma}_{\omega\mu\nu} = K_{\omega\mu\nu},$$

i.e.,

(1.1)b [3] $$2S_{\omega\mu\nu} - 4S_{\nu\beta\gamma}k_{[\omega}{}^{[\beta}k_{\mu]}{}^{\gamma]} - 4k_{[\omega}{}^{\beta}S_{\mu]\beta\gamma}k_{\nu}{}^{\gamma} = K_{\omega\mu\nu}$$

in singular cases. According to Theorems II (11.1), II (14.3) and II (17.1) we know that in the case of the first class a necessary and sufficient condition for the existence and uniqueness of the solution $S_{\omega\mu\nu}$ is $g \neq 0$; for the case of the second class this condition is $g(g - 2) \neq 0$; while in the case of the third class the solution $S_{\omega\mu\nu}$ of (1.1) always exists.

In this chapter we shall investigate solutions $S_{\omega\mu\nu}$ of (1.1), assuming (unless explicitly stated otherwise) that we have in the case of the first class

(1.2) $$g = 0$$

and in the case of the second class

(1.3)a $$g = 0$$

or

(1.3)b $$g = 2.$$

The third class is excluded in this chapter, for in this case the solution always exists.

It will turn out that a very useful tool for this investigation is the tensor first defined for the most general case by Nijenhuis, A., (1). In our case this tensor may be written as follows:

(1.4) $$\tfrac{1}{2}N_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=} (\partial_{\alpha}k_{[\lambda}{}^{\nu})k_{\mu]}{}^{\alpha} - k_{\alpha}{}^{\nu}\partial_{[\mu}k_{\lambda]}{}^{\alpha}.$$

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[1] The results of this chapter are taken from Hlavatý, V., (19).

[2] *Cf.* II (3.12).

[3] *Cf.* II (3.11).

Therefore, after the necessary prerequisites we shall investigate this tensor thoroughly, particularly its relationship to $S_{\lambda\mu}{}^{\nu}$. The corresponding results will serve later as an efficient basis for the solution of the main problems of this chapter. [1] Unless otherwise explicitly stated the results of the sections 1–6 hold for the first two classes — even in cases characterized by (1.2) and (1.3).

NOTE. We are entitled to use here the nonholonomic frame described in sections I (8), I (9), I (10), since the results of those sections were derived independently of the value of $g(g-2)$.

2. The tensor $N_{\lambda\mu}{}^{\nu}$.

THEOREM (2.1). *The nonholonomic components*

$$N_{xy}{}^{z} = \underset{x}{a}{}^{\lambda}\,\underset{y}{a}{}^{\mu}\,\overset{z}{a}{}_{\nu}\,N_{\lambda\mu}{}^{\nu} \tag{2.1}$$

of $N_{\lambda\mu}{}^{\nu}$ *are given by*

$$\tfrac{1}{2}N_{xy}{}^{z} = \{{}_{[x}{}^{z}{}_{y]}\}\,(\underset{z}{\lambda}-\underset{x}{\lambda})\,(\underset{z}{\lambda}-\underset{y}{\lambda}),\quad (xyz \neq) \tag{2.2)a}$$

$$N_{xy}{}^{y} = (\underset{y}{\lambda}-\underset{x}{\lambda})\partial_{x}\underset{y}{\lambda} = -\,N_{yx}{}^{y}. \tag{2.2)b}$$

PROOF. The symbol ∂ in (1.4) may be replaced by ∇, and the equation so obtained rewritten in nonholonomic components. Substituting into the latter equation from I (9.1), we obtain (2.2). This result is independent of any special value of the scalar g.

THEOREM (2.2). *Assume*

$$(\underset{z}{\lambda}-\underset{x}{\lambda})\,(\underset{z}{\lambda}-\underset{y}{\lambda}) \neq 0.\,{}^{2} \tag{2.3)a}$$

---

[1] In this chapter we shall use the following Roman indices:

i, j, k, p, q, r	from 1 to 4,
x, y, z, t	from 1 to 4,
a, b, c, d	from 1 to 2,
e, f, g, h	from 3 to 4.

The indices in the last three rows ($x, \ldots, h$) *do not* obey the usual summation convention.

[2] We have $\underset{z}{\lambda} = \underset{x}{\lambda}$ ($\underset{z}{\lambda} = \underset{y}{\lambda}$) if and only if $\underset{z}{\lambda} = \underset{x}{\lambda} = 0$ ($\underset{z}{\lambda} = \underset{y}{\lambda} = 0$). Hence, (2.3)a is automatically satisfied for the first class. (See Theorems I (6.1) and I (7.1).)

Then a necessary and sufficient condition for the basic vectors $\underset{x}{a}^\lambda$, $\underset{y}{a}^\lambda$ *to be* X_2*-forming is*

$$N_{xy}{}^{z} = 0 \quad (xyz \neq). \tag{2.3)b}$$

PROOF. A necessary and sufficient condition for $\underset{x}{a}^\nu$ and $\underset{y}{a}^\nu$ to be X_2-forming is

$$\underset{x}{a}^\lambda \underset{y}{a}^\mu \nabla_{[\mu} \overset{z}{a}_{\lambda]} = 0, \tag{2.4}$$

and this equation is equivalent to

$$\{{}_{[x}{}^{z}{}_{y]}\} = 0 \tag{2.5}$$

(*Cf.* II (13,1)). Equation (2.5) is by virtue of (2.3)a and (2.2)a equivalent to (2.3)b.

REMARK. This Theorem has been proved in Nijenhuis, A., (1) by another method.

THEOREM (2.3). *A necessary and sufficient condition for the basic values* $\underset{1}{\lambda}, \ldots \underset{4}{\lambda}$ *to be constant is*

$$N_{xy}{}^{y} = 0 \quad (x \neq y). \tag{2.6}$$

PROOF. Let us first introduce the indices x', y', z' by the following agreement:

If $x = 1, 2, 3, 4$, then $x' = 2, 1, 4, 3$, and similarly, for y', z'. Then we have by I (6.2) and I (7.2)

$$\partial_x \underset{x'}{\lambda} = - \partial_x \underset{x}{\lambda} \tag{2.7}$$

so that the condition

$$\partial_x \underset{y}{\lambda} = 0$$

for all x and all $y \neq x$ yields constant λ's. Then the Theorem follows at once for the first class from (2.2)b. If, however, we are dealing with the second class, then for $\underset{x}{\lambda} = \underset{y}{\lambda} = 0$, $x \neq y$, we also obtain (2.6) by virtue of (2.2)b.

The equations (2.3)b and (2.6) may be rewritten in a way which does not require the basic vectors.

THEOREM (2.4). *Let $u^\lambda (u_\lambda)$ be an arbitrary vector*[1]. *Equations* (2.3)b *and* (2.6) *are equivalent to*

$$u^\alpha u^\beta u_\gamma \underset{x}{E}_\alpha{}^\lambda \underset{y}{E}_\beta{}^\mu \overset{z}{E}_\nu{}^\gamma N_{\lambda\mu}{}^\nu = 0 \tag{2.8}$$

and

$$u^\alpha u^\beta u_\gamma \underset{x}{E}_\alpha{}^\lambda \underset{y}{E}_\beta{}^\mu \overset{y}{E}_\nu{}^\gamma N_{\lambda\mu}{}^\nu = 0. \tag{2.9}$$

The proof follows at once from Theorem I (10.3) and Remark II that follows it.

REMARK. If in (2.3)b $x, y = 3, 4$, or $x, y = 1, 2$, this equation may be rewritten in a way which does not involve the basic vectors and the arbitrary vectors, namely

$$\overset{xy}{Q}{}^{\lambda\mu} N_{\lambda\mu}{}^\alpha \overset{zt}{P}_\alpha{}^\nu = 0 \quad (xyzt \neq). \tag{2.10}$$

The proof follows easily from I (10.7) and I (10.8).

3. The tensors $N_{\lambda\mu}{}^\nu$ and $S_{\lambda\mu}{}^\nu$.

THEOREM (3.1). *Let D_μ be the symbol of the covariant derivative with respect to an arbitrary connection $\Gamma_\lambda{}^\nu{}_\mu$ whose skew symmetric part is $S_{\lambda\mu}{}^\nu$. Put*

$$T_{\lambda\mu}{}^\nu \overset{\text{def}}{=} D_\lambda k_\mu{}^\nu. \tag{3.1}$$

Then

$$\tfrac{1}{2} N_{\lambda\mu}{}^\nu = \tag{3.2}$$
$$= T_{\alpha[\lambda}{}^\nu k_{\mu]}{}^\alpha - T_{[\mu\lambda]}{}^\alpha k_\alpha{}^\nu - k_\beta{}^\nu (2S_{\alpha[\lambda}{}^\beta k_{\mu]}{}^\alpha + S_{\lambda\mu}{}^\alpha k_\alpha{}^\beta) - S_{\beta\alpha}{}^\nu k_{[\lambda}{}^\beta k_{\mu]}{}^\alpha.$$

PROOF. We have

$$T_{\xi\eta}{}^\zeta = \partial_\xi k_\eta{}^\zeta - S_{\eta\xi}{}^\alpha k_\alpha{}^\zeta + S_{\alpha\xi}{}^\zeta k_\eta{}^\alpha + (*) \tag{3.3}$$

where (*) denotes terms containing the symmetric part of the connection. Substituting $\partial_\xi k_\eta{}^\zeta$ from (3.3) into (1.4), we obtain (3.2).

The next auxiliary Theorem, needed later on, is

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[1] By an "arbitrary vector" we mean any vector whose transvection with $\overset{x}{E}_\lambda{}^\nu = \underset{x}{E}_\lambda{}^\nu$ ($x = 1, \ldots, 4$) is not equal to zero.

THEOREM (3.2). *If the connection* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ *is defined by* II (2.1), *then we have*

$$(3.4)\quad T_{\xi\eta}{}^{\zeta} = - S^{\zeta}{}_{\xi\alpha}(\delta^{\alpha}_{\eta} - {}^{(2)}k_{\eta}{}^{\alpha}) + (S_{\beta\xi}{}^{\alpha}k_{\eta}{}^{\beta} - S_{\xi\eta}{}^{\alpha})\,(\delta^{\zeta}_{\alpha} - k_{\alpha}{}^{\zeta}).$$

PROOF. Equation II (2.1)b yields in turn

$$(3.5)\text{a}\qquad D_{\omega}h_{\lambda\mu} = 2S_{\omega(\mu}{}^{\alpha}g_{\lambda)\alpha}$$

$$(3.5)\text{b}\qquad - h_{\lambda\mu}D_{\omega}h^{\lambda\nu} = 2S_{\omega(\mu}{}^{\alpha}g_{\lambda)\alpha}h^{\lambda\nu}$$

$$(3.5)\text{c}\qquad D_{\omega}h^{\lambda\nu} = - 2S_{\omega(\alpha}{}^{\gamma}g_{\beta)\gamma}h^{\beta\nu}h^{\alpha\lambda}$$

$$(3.5)\text{d}\qquad D_{\omega}k_{\lambda\mu} = 2S_{\omega[\mu}{}^{\alpha}g_{\lambda]\alpha}$$

$$(3.5)\text{e}\qquad D_{\xi}k_{\eta}{}^{\zeta} = (D_{\xi}k_{\eta\lambda})h^{\lambda\zeta} + k_{\eta\lambda}D_{\xi}h^{\lambda\zeta}.$$

Substituting into (3.5)e from (3.5)d and (3.5)c one obtains (3.4).

THEOREM (3.3). *If the connection* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ *is defined by* II (2.1), *then the tensors* $S_{\lambda\mu}{}^{\nu}$ *and* $N_{\lambda\mu}{}^{\nu}$ *are related by*

$$(3.6)\text{a}\qquad N_{\lambda\mu\nu} = 2A^{\alpha\beta\gamma}_{\lambda\mu\nu}S_{\alpha\beta\gamma}$$

where

$$(3.6)\text{b}^{1}\qquad \begin{aligned} A^{\xi\eta\zeta}_{\lambda\mu\nu} = & - \delta^{[\xi}_{\nu}k_{[\mu}{}^{\eta]}\delta^{\zeta}_{\lambda]} + {}^{(2)}k_{[\lambda}{}^{\zeta}k_{\mu]}{}^{[\eta}\delta^{\xi]}_{\nu} \\ & - \delta^{\zeta[\eta}_{\nu[\lambda}k_{\mu]}{}^{\xi]} - k_{\nu}{}^{\zeta}k_{[\lambda}{}^{[\eta}k_{\mu]}{}^{\xi]} \\ & - k_{\nu}{}^{[\xi}\delta^{\eta]\zeta}_{[\mu\ \lambda]} + {}^{(2)}k_{[\lambda}{}^{\zeta}\delta^{[\eta}_{\mu]}k_{\nu}{}^{\xi]} \\ & - k_{\nu}{}^{\zeta}\delta^{[\xi\eta]}_{[\mu\lambda]} - {}^{(2)}k_{\nu}{}^{\zeta}\delta^{[\xi}_{[\mu}k_{\lambda]}{}^{\eta]}. \end{aligned}$$

PROOF. One obtains (3.6) upon substituting into (3.2) from (3.4).

REMARK. There is a basic difference between (3.2), which holds in every case, and (3.6), which holds only in some cases.

THEOREM (3.4). *A sufficient condition for the relationship* (3.6) *to hold is that equation* II (2.1) *admit at least one solution* [2].

PROOF. The equations (3.6) result from II (2.1), (3.4), (1.4) and (3.2). The equations (1.4) and (3.2) hold in every case. The equation (3.4) exists only if II (2.1) is meaningful, *i.e.*, if it admits at least one solution.

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[1] There is an obvious misprint in the corresponding equation (3.6)b in Hlavatý, V., (19).

[2] A necessary and sufficient condition for II (2.1) to admit *exactly* one solution is $g \neq 0$ for the first class and $g(g - 2) \neq 0$ for the second class.

REMARK. This Theorem will play a paramount role in the degenerate case characterized by (1.2) and (1.3).

THEOREM (3.5). *The relation* (3.6) *is equivalent to*

$$N_{xyz} = S_{xyz}[\underset{x}{\lambda} + \underset{y}{\lambda} + \underset{z}{\lambda}\,\underset{xyz}{\lambda} + \underset{z}{\lambda}(1 + \underset{x}{\lambda}\underset{y}{\lambda})] + \tag{3.7}$$

$$+ S_{yzx}(\underset{y}{\lambda} + \underset{z}{\lambda})\,(1 - \underset{x}{\lambda^2}) + S_{zxy}(\underset{x}{\lambda} + \underset{z}{\lambda})\,(1 - \underset{y}{\lambda^2})$$

for arbitrary x, y, z.

PROOF. Equation (3.6)b shows that according to I (9.1) the nonholonomic components A^{ijk}_{pqr} for which ijk is not a permutation of pqr are equal to zero. Moreover, they are skew symmetric in ij as well as in pq. Therefore, the nonholonomic form of (3.6)a is

$$N_{xyz} = 2A^{ijk}_{xyz}S_{ijk} = \tag{3.8}$$

$$= 4A^{xyz}_{xyz}S_{xyz} + 4A^{yzx}_{xyz}S_{yzx} + 4A^{zxy}_{xyz}S_{zxy}.$$

Substituting into the nonholonomic components of A from I (9.1), one obtains from (3.8) the equation (3.7).

4. The identities.

Let us prove first an auxiliary

THEOREM (4.1). *The equations* (3.6) *cannot be inverted.*

PROOF. It is sufficient to prove that (3.7), equivalent to (3.6), cannot be inverted: For $z \neq x, x'$,[1] equation (3.7) yields

$$N_{xx'z} = [2S_{xx'z}\underset{z}{\lambda} + S_{x'zx}(\underset{z}{\lambda} - \underset{x}{\lambda}) + S_{zxx'}(\underset{z}{\lambda} + \underset{x}{\lambda})]\,\underset{xx'z}{\lambda} \tag{4.1)a}$$

$$N_{x'zx} = [S_{x'zx}(\underset{z}{\lambda} + \underset{x}{\lambda}) + S_{zxx'}(\underset{z}{\lambda} + \underset{x}{\lambda})]\,\underset{xx'z}{\lambda} \tag{4.1)b}$$

$$N_{zxx'} = [S_{x'zx}(\underset{z}{\lambda} - \underset{x}{\lambda}) + S_{zxx'}(\underset{z}{\lambda} - \underset{x}{\lambda})]\,\underset{xx'z}{\lambda}. \tag{4.1)c}$$

On the other hand, we obtain from (3.7)

$$N_{xyx} = 2\underset{x}{\lambda}\,\underset{xxy}{\lambda}\,S_{xyx}, \quad (x \neq y). \tag{4.2}$$

The equations (4.2) represent all equations for the nonholonomic components of N with two equal indices, while (4.1) represent all equations for the nonholonomic components of N with three different indices. The determinant of the system (4.1) — looked upon as a system for the unknowns S — is equal to zero. Hence, (4.1) cannot be inverted.

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[1] x' is defined in the proof of Theorem (2.3).

THEOREM (4.2). *The components* $N_{\lambda\mu}{}^{\nu}$ *are not independent. They satisfy the condition*

$$\overset{xy}{P}{}^{\alpha\beta}[k_{\lambda}{}^{\gamma}N_{\gamma\alpha\beta} + N_{\lambda\gamma(\alpha}k_{\beta)}{}^{\gamma}] = 0, \qquad xy = 12,\ 34 \tag{4.3}$$

and consequently, also

$$k_{\lambda}{}^{\gamma}N_{\gamma\alpha}{}^{\alpha} + N_{\lambda\gamma}{}^{\alpha}k_{\alpha}{}^{\gamma} = 0. \tag{4.4}$$

PROOF. Assuming first that $(g-2)\,g \neq 0$, we have by Theorems II (9.1)b, II (14.1)b $\underset{xx'z}{\lambda} \neq 0$. Then the last two equations (4.1) yield

$$\begin{aligned} 0 &= N_{x'zx}(\underset{z}{\lambda} - \underset{x}{\lambda}) - N_{zxx'}(\underset{z}{\lambda} + \underset{x}{\lambda}) = \\ &= -\underset{z}{\lambda}(N_{zx'x} + N_{zxx'}) - \underset{x'}{\lambda}N_{zx'x} - \underset{x}{\lambda}N_{zxx'}. \end{aligned} \tag{4.5}$$

This relation derived for $z \neq x, x'$ also holds for $z = x$ or $z = x'$ as a simple inspection shows. Hence, using I (9.1) we may rewrite (4.5) in the form

$$k_{i}{}^{j}N_{j(xx')} + N_{ij(x}k_{x')}{}^{j} = 0 \tag{4.6}$$

where the indices i, x are quite arbitrary. Hence, we have by virtue of I (10.7)

$$\overset{xx'}{P}{}^{pq}(k_{i}{}^{j}N_{jpq} + N_{ij(p}k_{q)}{}^{j}) = 0,$$

and this equation is equivalent to (4.3). Equation (4.4) is then an immediate consequence of (4.3) and I (10.7). The remaining part of the proof (for $(g-2)\,g = 0$) will be given with the proof of

THEOREM (4.3). *The conditions* (4.3) *and* (4.4) *do not impose any new restriction on* $k_{\lambda}{}^{\nu}$, *for these conditions are identically satisfied.*

PROOF. It is obviously sufficient to prove that (4.5) is an identity. Put

$$A(x, z) \overset{\text{def}}{=} (\underset{z}{\lambda} - \underset{x}{\lambda})N_{x'zx} - (\underset{z}{\lambda} + \underset{x}{\lambda})N_{zxx'}.$$

Then we have by virtue of I (8.5)

$$\Lambda(x, z) = (\underset{z}{\lambda} - \underset{x}{\lambda})N_{x'z}{}^{x'} - (\underset{z}{\lambda} + \underset{x}{\lambda})N_{zx}{}^{x},$$

or according to (2.2)b,

$$\begin{aligned}\Lambda(x, z) &= -(\underset{z}{\lambda} - \underset{x}{\lambda})(\underset{x'}{\lambda} - \underset{z}{\lambda})\partial_z \underset{x'}{\lambda} - (\underset{x}{\lambda} + \underset{z}{\lambda})(\underset{x}{\lambda} - \underset{z}{\lambda})\partial_z \underset{x}{\lambda} = \\ &= -(\underset{z}{\lambda} - \underset{x}{\lambda})(\underset{z}{\lambda} + \underset{x}{\lambda})\partial_z \underset{x}{\lambda} + (\underset{z}{\lambda} + \underset{x}{\lambda})(\underset{z}{\lambda} - \underset{x}{\lambda})\partial_z \underset{x}{\lambda} \equiv \\ &\equiv 0.\end{aligned}$$

Hence, (4.5) is an identity for any value of $g(g - 2)$.

5. The vector S_λ. In this section as in the next one, we shall deal only with *the case of the first class. Moreover, we shall also assume* $g \neq 0$.

One of the field equations of the unified theory is

$$S_\lambda \overset{\text{def}}{=} S_{\lambda\alpha}{}^{\alpha} = 0. \tag{5.1}$$

Throughout sections five and six we assume that (5.1) is satisfied.

THEOREM (5.1). *The condition* (5.1) *is equivalent to*

$$2\underset{z}{\lambda}\underset{zzz'}{\lambda}(S_{zxx'} + S_{zx'x}) = -N_{zz'z} \tag{5.2}$$

for all combinations of indices $z \neq x, x'$.

PROOF. Condition (5.1) is equivalent to

$$S_{zx}{}^{x} + S_{zx'}{}^{x'} + S_{zz'}{}^{z'} = 0,$$

and this equation may be written by virtue of I (8.5)

$$S_{zxx'} + S_{zx'x} + S_{zz'z} = 0. \tag{5.3}$$

On the other hand, we have from (4.2)

$$N_{zz'z} = 2\underset{z}{\lambda}\underset{zzz'}{\lambda}S_{zz'z}. \tag{5.4}$$

According to our assumptions we have $\underset{z}{\lambda}\underset{zzz'}{\lambda} \neq 0$. Therefore, we may substitute for $S_{zz'z}$ from (5.4) into (5.3) and, in this way, obtain (5.2).

THEOREM (5.2). *The equations* (3.6)a *and* (5.1) *admit a unique solution* $S_{\lambda\mu}{}^{\nu}$.

PROOF. Equations (3.6) are equivalent to (4.1) and (4.2), and condition (5.1) is equivalent to (5.2). If we replace (4.1)c by the latter equation, we obtain a system of three equations with the determinant

$$\Delta \overset{\text{def}}{=} 8(\underset{z}{\lambda}\ \underset{xx'z}{\lambda})^2\,(\underset{z}{\lambda} + \underset{x}{\lambda})\,\underset{zz'z}{\lambda} \neq 0.^{1} \tag{5.5}$$

Hence, this system yields S_{ijk} $(ijk \neq)$, while (4.2) gives the components S_{xyx}.

This Theorem will be needed in the next section.

6. Product space.

THEOREM (6.1). *A necessary and sufficient condition that*
a) *every pair of basic vectors of $k_\lambda{}^\nu$ be X_2-forming and*
b) *every basic value of $k_\lambda{}^\nu$ be constant is*

$$S_{\lambda\mu}{}^{\nu} = 0. \tag{6.1}$$

PROOF. Assume that (6.1) holds. Then (3.6)a yields

$$N_{\lambda\mu}{}^{\nu} = 0. \tag{6.2}$$

Hence, requirements a) and b) are satisfied by virtue of Theorems (2.2) and (2.3). Assume now that requirements a) and b) are satisfied so that (6.2) holds. Then the equations (4.1)a,b, (5.2) and (4.2), together with condition (5.5), yield (6.1).

In the next Theorem we denote by V_n an n-dimensional Riemann space.

THEOREM (6.2). *Let requirements a) and b) of the previous Theorem be satisfied. Then our space-time is a V_4, which is a product space, $V_4 = V_2 \times V_2$. There is a coordinate system e^x such that*

$$\tfrac{1}{2}ds^2 = h_{12}(e^1, e^2)de^1 de^2 + h_{34}(e^3, e^4)de^3 de^4, \tag{6.3}$$

and the connection Γ is given by

$$\Gamma_x{}^x{}_x = \{{}_x{}^x{}_x\} = \partial_x \ln h_{xx'} \tag{6.4a}$$

$$\textit{the remaining } \Gamma_x{}^z{}_y = 0. \tag{6.4b}$$

PROOF. Assume that requirement a) of the previous Theorem is satisfied. Then there is a coordinate system e^x in which the basic

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[1] We have $\Delta \neq 0$ for all possible x, $z \neq x$, x' because $k_{\lambda\mu}$ is of the first class and $g \neq 0$.

vectors satisfy the conditions

$$(6.5) \qquad \underset{j}{a}^{\nu} = \frac{\partial x^{\nu}}{\partial e^{j}}, \qquad \overset{i}{a}_{\lambda} = \frac{\partial e^{i}}{\partial x^{\lambda}}.$$

These basic vectors satisfy I (8.4), where

$$(6.6) \quad ((h_{ij})) = ((h_{ji})) = \left(\begin{pmatrix} 0 & h_{12} & 0 & 0 \\ h_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{34} \\ 0 & 0 & h_{43} & 0 \end{pmatrix}\right), \quad h_{12}h_{34} \neq 0.$$

Assume now that both requirements a) and b) of the previous Theorem hold. Then the basic values involved in I (9.1)b for $p = 1$, *i.e.*, in

$$(6.7) \qquad k_{xy} = \underset{x}{\lambda} h_{xy}$$

are constant. Moreover, from (6.1), II (3.2) and II (3.1) we have

$$(6.8) \qquad \Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\}.$$

If

$$(6.9)\text{a} \qquad K_{\omega\mu\nu} = 0$$ [1]

and $g \neq 0$, then we see from II (5.10) and II (3.1), II (3.2) that (6.1) holds and *vice versa*. Therefore, by virtue of II (3.14) and (6.1)

$$(6.9)\text{b} \qquad \nabla_{\omega} k_{\lambda\mu} = 0.$$

Consequently, by virtue of (6.6) and (6.7)

$$(6.10) \qquad 0 = \nabla_x k_{yz} + \underset{z}{\lambda} \nabla_x h_{yz} =$$

$$= \underset{y}{\lambda} \partial_x h_{yz} - \{{}_{y}{}^{z'}{}_{x}\} \underset{z'}{\lambda} h_{z'z} - \{{}_{z}{}^{y'}{}_{x}\} \underset{y}{\lambda} h_{yy'} +$$

$$+ \underset{z}{\lambda} (\partial_x h_{yz} - \{{}_{y}{}^{z'}{}_{x}\} h_{z'z} - \{{}_{z}{}^{y'}{}_{x}\} h_{yy'}) =$$

$$= (\underset{y}{\lambda} + \underset{z}{\lambda}) (\partial_x h_{yz} - \{{}_{z}{}^{y'}{}_{x}\} h_{yy'}).$$

On the other hand, we have

$$\underset{y}{\lambda} + \underset{z}{\lambda} \neq 0 \text{ for } z \neq y',$$

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[1] *Cf.* II (3.5).

so that according to (6.10)

$$\{{}_{x}{}^{y'}{}_{z}\} = \{{}_{z}{}^{y'}{}_{x}\} = 0 \text{ for } z \neq y'.\text{[1]} \tag{6.11}$$

Equation (6.11) yields for $y' = x$, $z \neq x$

$$\{{}_{z}{}^{x}{}_{x}\} = \{{}_{x}{}^{x}{}_{z}\} = 0. \tag{6.12}$$

On the other hand, according to (6.11), for $z = y'$, $z \neq x$

$$\{{}_{z}{}^{z}{}_{x}\} = \{{}_{z}{}^{y'}{}_{x}\} = \{{}_{x}{}^{y'}{}_{z}\} = 0. \tag{6.13}$$

Therefore, the usual definition of Christoffel symbols leads by virtue of (6.12) and (6.6) to

$$\begin{aligned} 0 = \{{}_{x}{}^{x}{}_{z}\} &= \tfrac{1}{2}h^{x\iota}(\partial_x h_{\iota z} + \partial_z h_{\iota x} - \partial_\iota h_{xz}) \\ &= \tfrac{1}{2}h^{xx'}\partial_z h_{xx'} = \tfrac{1}{2}\partial_z \ln h_{xx'} \\ &\qquad z \neq x, x', \end{aligned} \tag{6.14}$$

so that

$$h_{xx'} = h_{xx'}(e^x, e^{x'}). \tag{6.15}$$

Moreover, the same definition of Christoffel symbols yields

$$\{{}_{x}{}^{x}{}_{x'}\} = \{{}_{x'}{}^{x}{}_{x}\} = 0\,\text{[2]}, \quad \{{}_{x}{}^{x}{}_{x}\} = \partial_x \ln h_{xx'}. \tag{6.16}$$

The equations (6.8), (6.11), (6.12), (6.13) and (6.16) lead to (6.4), while equation (6.3) results from (6.6) and (6.15).

In the next Theorem we denote by E_2 (M_2) a surface of zero Gaussian curvature and Euclidean (Minkowskian) metric.

THEOREM (6.3). *There is a factor of proportionality ρ such that for*

$$'h_{\lambda\mu} = \rho h_{\lambda\mu}$$

the constituents of the corresponding V_4 are E_2 and M_2.

PROOF. The factor $\rho \overset{\text{def}}{=} h^{12}h^{34}$ reduces ds^2 involved in (6.3) to

$$\tfrac{1}{2}\rho ds^2 = h^{34}(e^3, e^4)de^1 de^2 + h^{12}(e^1, e^2)de^3 de^4.$$

---

[1] One must bear in mind that in our case, by (2.5),

$$\{{}_{[x}{}^{y'}{}_{y]}\} = 0.$$

Furthermore, we have $\underset{z}{\lambda} + \underset{y}{\lambda} \neq 0$, $h_{zy} = 0$ for $y' \neq z$.

[2] This equation also results from (6.12).

Hence, if e^1, e^2, (e^3, e^4) are constant, the corresponding V_2-metric is of zero curvature. Two of the basic vectors being complex-conjugate and the remaining two real, the surfaces mentioned above reduce to E_2 and M_2.

PART II

Degenerate cases

7. First class. In this second part we shall deal with solutions of (1.1) in exceptional cases characterized by (1.2) and (1.3). In this section we consider the exceptional case of the first class characterized by

$$k \neq 0 \tag{7.1)a}$$

$$g = 1 + 2K + k = 0. \tag{7.1)b}$$

LEMMA (7.1). *We have*[1]

$$\alpha \overset{\text{def}}{=} \underset{1}{\lambda} = -\underset{2}{\lambda} = \sqrt{k}, \tag{7.2)a}$$

$$\beta \overset{\text{def}}{=} \underset{3}{\lambda} = -\underset{4}{\lambda} = 1, \tag{7.2)b}$$

and

(7.3)a[2]
$$\underset{34i}{\lambda} = 0,$$

$$\underset{12i}{\lambda} = 1 - k, \tag{7.3)b}$$

$$\underset{aff}{\lambda} = 2(1 + \varepsilon\sqrt{k}), \tag{7.3)c}$$

$$\underset{aaf}{\lambda} = (1 + \varepsilon\sqrt{k})^2, \tag{7.3)d}$$

$$(\varepsilon = (-1)^{a+f}).$$

PROOF. The equations (7.2) follow at once from II (9.1) and (7.1). The equations (7.3) follow by simple inspection from II (9.2) and (7.2).

In order to prove the next Theorems we need the following two Lemmas, which will be proved simultaneously.

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[1] For the first class we have $k < 0$.

[2] See the footnote 1 on p. 104.

LEMMA (7.2). *We have*

$$\nabla_i k_{fe} = 0 \tag*{(7.4)}$$

and

$$\{{}_{[e}{}^{a'}{}_{f]}\} + \tfrac{1}{2}\{{}_{a}{}^{e'}{}_{f}\} - \tfrac{1}{2}\{{}_{a}{}^{f'}{}_{e}\} = 0 \tag*{(7.5)a}$$

$$\{{}_{(e}{}^{a'}{}_{f)}\} + \tfrac{1}{2}\{{}_{a}{}^{e'}{}_{f}\} + \tfrac{1}{2}\{{}_{a}{}^{f'}{}_{e}\} = 0. \tag*{(7.5)b}$$

LEMMA (7.3). *The following equation holds:*

$$K_{ief} = K_{i(ef)}. \tag*{(7.5)c}$$

PROOF. First of all, from I (9.1) we have for $e \neq f$

$$k_{ef} = \underset{e}{\lambda} h_{ef} (= - \underset{f}{\lambda} h_{ef})$$

so that

$$\begin{aligned} \nabla_i k_{ef} &= \partial_i k_{ef} - \{{}_{e}{}^{f'}{}_{i}\} k_{f'f} - \{{}_{f}{}^{e'}{}_{i}\} k_{ee'} = \\ &= \underset{e}{\lambda}(\partial_i h_{ef} - \{{}_{e}{}^{f'}{}_{i}\} h_{f'f} - \{{}_{f}{}^{e'}{}_{i}\} h_{ee'}) \\ &= \underset{e}{\lambda}(\nabla_i h_{ef}) = 0. \end{aligned}$$

Moreover,

$$0 = 2\nabla_{[f} h_{e]a} = -2[-\{{}_{[f}{}^{a'}{}_{e]}\} + \tfrac{1}{2}\{{}_{a}{}^{e'}{}_{f}\} - \tfrac{1}{2}\{{}_{a}{}^{f'}{}_{e}\}]$$

and

$$0 = 2\nabla_{(f} h_{e)a} = -2[\{{}_{(f}{}^{a'}{}_{e)}\} + \tfrac{1}{2}\{{}_{a}{}^{e'}{}_{f}\} + \tfrac{1}{2}\{{}_{a}{}^{f'}{}_{e}\}].$$

The equation (7.5)c follows at once from (7.4) and II (3.5).

The following two Theorems will be proved simultaneously.

THEOREM (7.1)a. *The following equation holds:*

$$k_{(\omega}{}^{\nu} K_{\mu)\lambda\nu} \overset{34}{P}{}^{\omega\mu} = 0;^{1} \tag*{(7.6)}$$

it is equivalent to

$$K_{\lambda\omega\mu} \overset{34}{Q}{}^{\omega\mu} = 0. \tag*{(7.7)}$$

THEOREM (7.1)b. *A necessary and sufficient condition for* (1.1) *to admit at least one solution* $S_{\alpha\beta\gamma}$ *is*

$$k_{(\lambda}{}^{\nu} K_{\omega)\mu\nu} \overset{34}{Q}{}^{\omega\mu} = 0. \tag*{(7.8)}$$

---

[1] *Cf.* I (10.6).

PROOF. Since the scalar k is negative, it follows from (7.3) that the only scalars $\underset{ijk}{\lambda}$ which are equal to zero are $\underset{34i}{\lambda}$. Let us now consider the system II (11.3) for $x = e$, $y = f \neq e$, $z = a$. This system cannot be solved uniquely for the S's, its determinant $\underset{34a}{\Delta}$ being equal to zero according to the equation II (9.12). This system reduces by virtue of (7.3)a to

$$\begin{aligned} \tfrac{1}{2}K_{efa} &= - S_{efa}\underset{e}{\lambda}\underset{f}{\lambda} - S_{fae}\underset{e}{\lambda}\underset{f}{\lambda} - S_{aef}\underset{e}{\lambda}\underset{f}{\lambda} \\ \tfrac{1}{2}K_{fae} &= - S_{efa}\underset{f}{\lambda}\underset{a}{\lambda} - S_{fae}\underset{f}{\lambda}\underset{a}{\lambda} - S_{aef}\underset{f}{\lambda}\underset{a}{\lambda} \\ \tfrac{1}{2}K_{aef} &= - S_{efa}\underset{a}{\lambda}\underset{e}{\lambda} - S_{fae}\underset{a}{\lambda}\underset{e}{\lambda} - S_{aef}\underset{a}{\lambda}\underset{e}{\lambda} \end{aligned} \tag{7.9}$$

so that a necessary and sufficient condition for (7.9) to admit at least one solution is obviously

$$\underset{a}{\lambda}K_{efa} = \underset{e}{\lambda}K_{fae} = \underset{f}{\lambda}K_{aef}. \tag{7.10)a}$$

On the other hand, the left-hand side of the first equation of II (11.3) vanishes for $x = e$, $y = f$, $z = f$. Hence, we must also have

$$K_{eff} = 0. \tag{7.10)b}$$

Equations (7.10)a,b can be condensed into

$$K_{efi}\underset{i}{\lambda} = K_{fie}\underset{e}{\lambda} = K_{ief}\underset{f}{\lambda}, \quad \text{(no sum for } i\text{)}. \tag{7.10)c}$$

Some of these conditions are satisfied as we shall see later.

First let us consider the last of equations (7.10)c which, by virtue of I (9.1), may be written as follows:

$$\begin{aligned} 0 &= K_{fij}k_e{}^j - K_{iej}k_f{}^j \\ &= 2k_{(e}{}^jK_{f)ij}. \end{aligned} \tag{7.11)a}$$

Moreover, we may rewrite the same equation in another form resulting from (7.2)b:

$$0 = K_{fie} + K_{ief} = 2K_{i[ef]}. \tag{7.11)b}$$

Remembering I (8.1), I (8.3), I (10.6), I (10.7) and I (10.8) we see that (7.11)a,b are equivalent to (7.6) and (7.7). On the other hand, (7.11)b is satisfied by virtue of (7.5)c.

From (7.10)c we have also

$$0 = K_{efi}\underset{i}{\lambda} - \tfrac{1}{2}(K_{fie}\underset{e}{\lambda} + K_{ief}\underset{f}{\lambda})$$

$$= K_{efj}k_i{}^j - k_{[e}{}^j K_{f]ij}, \text{[1]}$$

or, by virtue of I (10.8)d,

$$0 = \overset{34}{Q}{}^{\omega\mu}(K_{\omega\mu\nu}k_\lambda{}^\nu - k_\omega{}^\nu K_{\mu\lambda\nu})$$

$$= 2\overset{34}{Q}{}^{\omega\mu}k_{(\lambda}{}^\nu K_{\omega)\mu\nu}.$$

A simple inspection shows that equations (7.6), (7.7) and (7.8) are equivalent to (7.10)c.

REMARK. According to II (3.2) we have

$$U^\nu{}_{\lambda\mu} \stackrel{\text{def}}{=} 2h^{\alpha\nu}S_{\alpha(\lambda}{}^\beta k_{\mu)\beta}.$$

In Chapter II (Corollary (7.1)) we found that a necessary and sufficient condition for $U^\nu{}_{\lambda\mu} = 0$ is

$$k_{(\lambda}{}^\nu K_{\omega)\mu\nu} = 0.$$

If this condition is satisfied, then the conditions (7.6) and (7.8) are identically satisfied. However, we cannot use this result here, for the necessary and sufficient condition mentioned above was derived under the assumption $g \neq 0$ which contradicts (7.1)b.

THEOREM (7.2). *Whenever* (7.8) *holds, the basic vectors* $\underset{3}{a}{}^\nu$, $\underset{4}{a}{}^\nu$ *are* X_2*-forming.*

PROOF. Whenever (7.8) holds, the system (1.1) admits at least one solution. Therefore, by virtue of II (3.1), the system II (2.1) admits at least one solution, and consequently, according to Theorem (3.4), the relationship (3.6) exists. Therefore, the system (4.1) exists. For $x = e$, $x' = f \neq e$, $z = a$ this system reduces according to (7.3) to

(7.12)a $$N_{ef}{}^{a'} = 0$$

(7.12)b $$N_{af}{}^{f} = 0$$

(7.12)c $$N_{ae}{}^{e} = 0.$$

The equations (7.12)b,c confirm the fact that $\underset{3}{\lambda}$, $\underset{4}{\lambda}$ are constant (*Cf.*

---

[1] No sum for i.

equations (2.2)b). Equation (7.12)a asserts, by virtue of Theorem (2.2), that the basic vectors $\underset{3}{a}^{\nu}$, $\underset{4}{a}^{\nu}$ are X_2-forming, *i.e.*,

$$\{_{[e}{}^{a}{}_{f]}\} = 0.\ ^{1} \tag{7.13}$$

THEOREM (7.3). *A necessary and sufficient condition for the system* (1.1) *to admit at least one solution is that the basic vectors* $\underset{3}{a}^{\nu}$, $\underset{4}{a}^{\nu}$ *be* X_2*-forming.*

PROOF. The necessity follows at once from Theorems (7.1)b and (7.2). Assume now that $\underset{3}{a}^{\nu}$, $\underset{4}{a}^{\nu}$ are X_2-forming, so that (7.13) holds. Therefore, the equations (7.5)a,b reduce to

$$\{_{a}{}^{f}{}_{f}\} = \{_{a}{}^{e}{}_{e}\} \tag{7.14)a$$

$$2\{_{e}{}^{a'}{}_{f}\} + \{_{a}{}^{f}{}_{f}\} + \{_{a}{}^{e}{}_{e}\} = 0 \tag{7.14)b$$

so that according to (7.4), (6.7), (7.13) and (7.2)

$$\underset{a}{\lambda}K_{efa} - \underset{e}{\lambda}K_{fae} = \tag{7.15}$$

$$= -\underset{e}{\lambda}\{\underset{a}{\lambda}[2\{_{e}{}^{a'}{}_{f}\} + \{_{a}{}^{e}{}_{e}\} + \{_{a}{}^{f}{}_{f}\}] + \underset{e}{\lambda}[\{_{a}{}^{e}{}_{e}\} - \{_{a}{}^{f}{}_{f}\}]\} = 0.$$

Hence, the first equation (7.10)c for $i = a$ is satisfied. The remaining equations (7.10)c are satisfied by virtue of Theorem (7.1)a (equation (7.7)).

8. First class. Continuation.

THEOREM (8.1). *Let* $k_{\mu\lambda}$ *be of the first class, and assume that* (7.1) *holds and that the basic vectors* $\underset{3}{a}^{\lambda}$, $\underset{4}{a}^{\nu}$ *are* X_2*-forming. Then the system* II (2.1) *admits at least one solution* II (3.1). *Its skew symmetric part* $S_{\lambda\mu}{}^{\nu}$ *has the following nonholonomic components:*

a) *The components* S_{eff} *are arbitrary* ($e \neq f$).

b) *The components* S_{efa}, S_{fae}, S_{aef} *are arbitrary except for the*

---

[1] As a matter of fact we have $k \neq 1$, and therefore, (7.12)a is equivalent to

$$\{_{[e}{}^{a'}{}_{f]}\} = 0,$$

and this equation is equivalent to (7.13), since the range of a and a' is the same, namely 1, 2.

single condition

$$\tfrac{1}{2}K_{efa} = 3S_{[efa]}.\qquad (8.1)$$ [1]

c) *The remaining components are*

$$2S_{fab} = \frac{1}{\underset{fab}{\lambda}}\,[K_{fab} + 3\underset{f}{\lambda}\underset{a}{\lambda}K_{[fab]}] \qquad (8.2)\text{a}$$

$$2S_{abf} = \frac{1}{\underset{fab}{\lambda}}\,[K_{abf} + 3\underset{a}{\lambda}\underset{b}{\lambda}K_{[abf]}] \qquad (8.2)\text{b}$$

$$2S_{abc} = \frac{1}{\underset{abc}{\lambda}}\,K_{abc} \qquad (8.2)\text{c}$$

$$2S_{eae} = \frac{1}{\underset{aee}{\lambda}}\,K_{eae} \qquad (8.2)\text{d}$$

where the λ*'s are to be taken from* (7.2) *and* (7.3).

PROOF. According to previous theorems, the system (1.1) admits at least one solution. The left-hand side of the first equation II (11.3) vanishes for $x = e$, $y = f = z$ so that this equation does not yield S_{eff}. There is no other equation among II (11.3) yielding S_{eff}. Hence, S_{eff} are not defined by II (11.3). Equation (8.1) follows at once from the first equation of (7.9). The remaining components given by (8.2) result from II (11.5)a.

REMARK. Put $x = f$, $y = e \neq f$ in (4.2):

$$N_{fef} = N_{fe}{}^{e} = -\,2\underset{f}{\lambda}\,\underset{ffe}{\lambda}\,S_{eff}.$$

The left term vanishes for $\underset{3}{\lambda} = -\underset{4}{\lambda} = \text{const.}$ (*Cf.* equations (2.2)b). The right term vanishes, since $\underset{ffe}{\lambda} = 0$. Hence, not even (4.2) can be used for finding S_{eff}.

In the non-degenerate case, equation (5.1) imposes some con-

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[1] Using (7.2)b, (7.3)a and (7.10)c we easily obtain for $e \neq f$

$$3K_{[efi]} = K_{efi}\left(1 + \frac{\underset{i}{\lambda}}{\underset{f}{\lambda}} + \frac{\underset{i}{\lambda}}{\underset{e}{\lambda}}\right) = \frac{\underset{efi}{\lambda} - 1}{\underset{f}{\lambda}\underset{e}{\lambda}}\,K_{efi} = K_{efi}$$

(no sum for i).

ditions on $g_{\lambda\mu}$. In the degenerate case the requirement (5.1) does not impose any condition on $g_{\lambda\mu}$:

THEOREM (8.2). *There is a two-parameter set of solutions* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, *mentioned in the previous Theorem, which satisfy* (5.1). *Its skew symmetric part* $S_{\omega\mu}{}^{\nu}$ *has the following nonholonomic components:*

a) *The components* S_{eff} *are given by*

$$(8.3)\text{a} \qquad S_{eff} = \frac{1}{1-k}\,[K_{f(ab)} + 3\underset{a}{\lambda}\,\underset{f}{\lambda}K_{[fab]}], \quad a \neq b, \ e \neq f.$$

b) *Two components, namely* S_{efa}, *are arbitrary while*

$$(8.3)\text{b} \qquad 2S_{aef} = \tfrac{1}{2}K_{efa} - \frac{1}{2(1-k)}\,K_{aba} - S_{efa}.$$

c) *The remaining components are given by* (8.2).

PROOF. Equation (5.1) is equivalent to

$$(8.4)\text{a} \qquad S_{fef} + S_{fab} + S_{fba} = 0$$

$$(8.4)\text{b} \qquad S_{aba} + S_{afe} + S_{aef} = 0 \qquad (a \neq b, e \neq f).$$

From (8.4)a and (8.2)a we obtain for $a \neq b$, $e \neq f$,

$$\begin{aligned} S_{eff} &= \frac{1}{2\underset{fab}{\lambda}}\,[2K_{f(ab)} + 3\underset{a}{\lambda}\,\underset{f}{\lambda}K_{[fab]} + 3\underset{b}{\lambda}\,\underset{f}{\lambda}K_{[fba]}] \\ &= \frac{1}{2(1-k)}\,[2K_{f(ab)} + 6\underset{a}{\lambda}\,\underset{f}{\lambda}K_{[fab]}]. \end{aligned}$$

On the other hand, (8.4)b and (8.1) yield

$$2S_{aef} = \tfrac{1}{2}K_{efa} - S_{aba} - S_{efa}, \qquad (a \neq b, \ e \neq f).$$

Substituting from (8.2)c into this equation one obtains (8.3)b.

9. Second class: The first case. The second class is characterized by the condition

$$K \neq 0, \quad k = 0.$$ [1]

There are two degenerate cases, namely

$$(9.1)\text{a} \qquad g = 0,$$

$$(9.1)\text{b} \qquad g = 2.$$

---

[1] *Cf.* II (14.6).

In this section we consider the case (9.1)a and prove first the following

LEMMA (9.1). *We have*

(9.2)a $$\alpha = \underset{1}{\lambda} = -\underset{2}{\lambda} = 0$$

(9.2)b $$\beta = \underset{3}{\lambda} = -\underset{4}{\lambda} = 1$$

and

(9.3)a $$\underset{34i}{\lambda} = 0$$

(9.3)b $$\underset{12i}{\lambda} = \underset{aaf}{\lambda} = 1$$

(9.3)c $$\underset{aff}{\lambda} = 2.$$

PROOF. Equations (7.2), (7.3) hold for any value of k which satisfies (7.1)b, (*i.e.*, (9.1)a). Hence, they also hold for $k = 0$ and reduce, in this case, to (9.2) and (9.3).

The following two Theorems will be proved simultaneously.

THEOREM (9.1)a. *We have*

(9.4)a $$K_{\omega\mu\nu}\overset{34}{Q}{}^{\omega\mu}\overset{34}{P}{}^{\lambda\nu} = 0.$$

THEOREM (9.1)b. *A necessary and sufficient condition for* (1.1) *to admit at least one solution* $S_{\lambda\mu}{}^{\nu}$ *is*

(9.4)b $$K_{\omega\mu\lambda}\overset{34}{P}{}^{\mu\lambda}\overset{12}{P}{}^{\omega\nu} = 0.$$

PROOF. A simple inspection shows that the required necessary and sufficient conditions may be written in the form (7.10)a for $\underset{a}{\lambda} = 0$ and (7.10)b so that by virtue of (7.4) the equations (7.10)c are equivalent to

(9.5) $$0 = K_{ief} = -2\nabla_{(e}k_{f)i} = K_{i(ef)}, \quad e \neq f.$$

For $i = f$ we obtain from (9.5) equation (7.10)b which is satisfied because of (7.4). On the other hand, (7.10)b may be written in the form (9.4)a. Hence, the remaining necessary and sufficient conditions reduce to (9.5) for $i = a$, and this system is obviously equivalent to (9.4)b.

THEOREM (9.2). *A necessary and sufficient condition that* (1.1) *admit at least one solution* $S_{\lambda\mu}{}^{\nu}$ *is that the basic vectors* $\underset{3}{a}{}^{\nu}$, $\underset{4}{a}{}^{\nu}$ *be* X_2-*forming.*

The proof of necessity follows the same pattern as the proof of Theorem (7.2). Assume now that $\underset{3}{a}{}^{\nu}$, $\underset{4}{a}{}^{\nu}$ are X_2-forming so that (7.13) holds. Therefore, from (6.7), (7.4), (7.5)a, (7.14)a and (9.2)a we derive for $e \neq f$

$$\begin{aligned} K_{fae} &= 2\nabla_{(f}k_{e)a} = \\ &= -[\{a^f{}_f\}k_{ef} + \{a^e{}_e\}k_{fe}] \\ &= -\,k_{ef}[\{a^f{}_f\} - \{a^e{}_e\}] = 0. \end{aligned}$$

These equations together with (7.10)b are condensed in the first equation (9.5).

THEOREM (9.3). *Let* $k_{\mu\lambda}$ *be of the second class, and assume that* (9.1)a *holds and that the basic vectors* $\underset{3}{a}{}^{\nu}$, $\underset{4}{a}{}^{\nu}$, *are* X_2-*forming. Then the system* II (2.1) *admits at least one solution* II (3.1). *Its skew symmetric part* $S_{\lambda\mu}{}^{\nu}$ *has the following nonholonomic components*:

a) *The components* S_{eff} *are arbitrary* ($e \neq f$).

b) *The components* S_{efa}, S_{fae}, S_{aef} *are arbitrary except for the single condition*

$$\tag{9.6} \tfrac{1}{2}K_{efa} = 3S_{[efa]}.$$[1]

c) *The remaining components are*

$$\tag{9.7)a} 2S_{fab} = K_{fab}$$

$$\tag{9.7)b} 2S_{abf} = K_{abf}$$

$$\tag{9.7)c} S_{abc} = 0$$

$$\tag{9.7)d} 2S_{eae} = \tfrac{1}{2}K_{eae}.$$

The proof follows the pattern of the proof of Theorem (8.1). As far as (9.7)c is concerned one must keep in mind that, according to I (9.1), I (8.5) and (9.2), the only component k_{ij} not equal to zero is $k_{34} = -\,k_{43}$. Hence, we have

$$\nabla_a k_{bc} = 0,$$

---

[1] From (9.5) we have

$$K_{efa} = 3K_{[efa]}.$$

and therefore, also $K_{abc} = 0$. This equation, together with (8.2)c and (9.3)b, leads to (9.7)c.

THEOREM (9.4). *There is a two-parameter set of solutions* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, *mentioned in the previous Theorem, which satisfy* (5.1). *Its skew symmetric part* $S_{\lambda\mu}{}^{\nu}$ *has the following components:*

a) *The components* S_{eff} *are given by*

$$S_{eff} = K_{f(ab)}; \quad a \neq b, \ e \neq f. \tag*{(9.8)a}$$

b) *The two components* S_{efa} *are arbitrary while*

$$2S_{aef} = \tfrac{1}{2}K_{efa} - S_{efa}. \tag*{(9.8)b}$$

c) *The remaining components are given by* (9.7).

The proof follows the pattern of the proof of Theorem (8.2).

10. Second class: The second case.

The second degenerate case of the second class is characterized by the conditions

$$K \neq 0, \quad k = 0$$ [1]

and (9.1)b, *i.e.*,

$$g = 2. \tag*{(10.1)}$$

In this section we shall consider this case.

LEMMA (10.1). *We have*

$$\alpha \overset{\text{def}}{=} \underset{1}{\lambda} = -\underset{2}{\lambda} = i \tag*{(10.2)a}$$

$$\beta \overset{\text{def}}{=} \underset{3}{\lambda} = -\underset{4}{\lambda} = 0 \tag*{(10.2)b}$$

and

$$\underset{12i}{\lambda} = 2 \tag*{(10.3)a}$$

$$\underset{34i}{\lambda} = \underset{aff}{\lambda} = 1 \tag*{(10.3)b}$$

$$\underset{aaf}{\lambda} = 0. \tag*{(10.3)c}$$

PROOF. Equations (10.2) follow at once from II (14.1) and (10.1), because the last of these equations is equivalent to $2K = 1$. The remaining equations follow from II (14.3) and (10.1).

---

[1] *Cf.* II (14.6).

THEOREM (10.1). *A necessary and sufficient condition for the system* (1.1) *to admit at least one solution* $S_{\lambda\mu}{}^{\nu}$ *is*

$$K_{faa} = 0. \tag{10.4}$$

PROOF. According to (10.3) and II (9.12), the determinant $\left(\underset{xyz}{\lambda}\right)^2$ of II (11.3) (for $xyz \neq$) is different from zero. Hence, when $xyz \neq$, this system does not impose any condition on $k_{\lambda\mu}$. Moreover, according to (10.2), (10.3) and II (9.12) it is obvious that the only possible restriction imposed upon $k_{\lambda\mu}$ by II (11.3) might result from the first of equations II (11.3) for $x = f, y = z = a$.[1] This restriction is (10.4), which obviously is not only necessary but also sufficient for the system II (11.3) (equivalent to (1.1)) to admit at least one solution.

In the next Theorem we denote by the name "the trajectories of $\underset{a}{a}^{\nu}$" the set of curves with the tangential vector field $\underset{a}{a}^{\nu}$.

THEOREM (10.2). *A necessary and sufficient condition for the system* (1.1) *to admit at least one solution is that the trajectories of* $\underset{a}{a}^{\nu}$ *be geodesic lines.*

PROOF. First of all we have (for $e \neq f$)

$$\begin{aligned} K_{faa} &= K_{faa} + 2\underset{a}{\lambda}\nabla_a h_{fa} = \\ &= 2[\nabla_a k_{fa} + \underset{a}{\lambda}\nabla_a h_{fa}] = -\, 2\underset{a}{\lambda}\{{}_a{}^e{}_a\}. \end{aligned} \tag{10.5}$$

Hence, we see from II (13.1)a that (10.4) is equivalent to

$$\underset{a}{a}^{\mu}\underset{a}{a}^{\lambda}\nabla_{\mu}\overset{e}{a}_{\lambda} = 0. \tag{10.6}$$

Assume now that the trajectories of $\underset{a}{a}^{\nu}$ are geodesic lines

$$\underset{a}{a}^{\lambda}\nabla_{\lambda}\underset{a}{a}^{\nu} = \underset{aa}{\Psi}\underset{a}{a}^{\nu}. \tag{10.7}$$

Then (10.6) is satisfied, and therefore, by virtue of (10.5), (10.4) is also satisfied. Conversely, assume now that (10.4) is satisfied. Then we see from (10.5) that (10.6) holds, and this condition is equivalent to

$$\underset{a}{a}^{\lambda}\nabla_{\lambda}\underset{a}{a}^{\nu} = \underset{aa}{\Psi}\,\underset{a}{a}^{\nu} + \underset{ab}{\Psi}\,\underset{b}{a}^{\nu} \quad (a \neq b), \tag{10.8}$$

---

[1] The second equation II (11.3) reduces to the identity 0 = 0, while the last equation reduces to the first one.

so that if we multiply (10.8) by $\overset{b}{a}_\nu$, we obtain by virtue of I (8.1)a and I (8.16)b

$$\begin{aligned} \underset{ab}{\Psi}\,\underset{b}{a}^\nu\overset{b}{a}_\nu = \underset{ab}{\Psi} &= \underset{a}{a}^\lambda(\nabla_\lambda \underset{a}{a}^\nu)\overset{b}{a}_\nu = \\ &= \underset{a}{a}^\lambda(\nabla_\lambda \underset{a}{a}^\nu)\underset{a}{a}_\nu \qquad = \\ &= \underset{a}{a}^\lambda(\nabla_\lambda \underset{a}{a}^\mu)\underset{a}{a}^\nu h_{\mu\nu}. \end{aligned}$$

The last term is obviously equal to zero, for $\underset{a}{a}^\nu$ is a null vector. Hence, we have $\underset{ab}{\Psi} = 0$, and (10.8) reduces to (10.7) (for $\underset{aa}{\Psi} = \underset{a}{\Psi}$).

THEOREM (10.3). *A sufficient condition for the system* (1.1) *to admit at least one solution is that the basic vectors* $\underset{a}{a}^\nu$ *and* $\underset{f}{a}^\nu$ *be* X_2-*forming* ($a = 1$ *or* 2, $f = 3$ *or* 4).

PROOF. Assume that $\underset{a}{a}^\nu$ and $\underset{f}{a}^\nu$ are X_2-forming:

$$\{{}_{[a}{}^u{}_{f]}\} = 0 \quad u \neq a, f. \tag{10.9)a (}$$

It is easily seen that for $b \neq a$

$$\begin{aligned} K_{faa} &= K_{faa} + \underset{a}{\lambda}\nabla_f h_{aa} \\ &= 4\underset{a}{\lambda}\{{}_{[f}{}^b{}_{a]}\}. \end{aligned} \tag{10.9)b (}$$

Hence, if (10.9)a is satisfied, so is (10.4).

REMARK. If we substitute into (4.2) $x = a$, $y = f$, and then $x = f$, $y = a$, we obtain, according to (10.3)c and (10.2)b,

$$\left.\begin{array}{ll} (10.9)\mathrm{c}_1 \qquad & N_{af}{}^b = 0, \\ (10.9)\mathrm{c}_2 \qquad & N_{fa}{}^e = 0, \end{array}\right. \quad a \neq b,\ e \neq f.$$

However, these equations are not sufficient for $\underset{f}{a}^\nu$ and $\underset{a}{a}^\nu$ to be X_2-forming, since condition (2.3)a is not satisfied for $z = e$, $x = f$, $y = a$.

THEOREM (10.4). *Let* $k_{\lambda\mu}$ *be of the second class, and assume that* (10.1) *holds and that the trajectories of* $\underset{a}{a}^\nu$ *are geodesic lines. Then the* *system* II (2.1) *admits at least one solution* $\Gamma_\lambda{}^\nu{}_\mu$. *Its skew symmetric part* $S_{\lambda\mu}{}^\nu$ *has the following nonholonomic components:*

a) *The four components* S_{faa} *are arbitrary.*

b) *The remaining components are*

$$S_{efg} = S_{abc} = 0 \tag{10.10)a}$$

$$2S_{efa} = K_{efa} \tag{10.10)b}$$

$$2S_{abf} = \tfrac{1}{2}(K_{abf} + 3K_{[abf]}), \qquad (a \neq b,\ e \neq f) \tag{10.10)c}$$

$$2S_{afb} = \tfrac{1}{2}K_{afb} \tag{10.10)d}$$

$$2S_{afg} = K_{afg}. \tag{10.10)e}$$

PROOF. Here we have

$$\nabla_i\, k_{ef} = 0, \quad \nabla_j\, k_{ab} = \underset{a}{\lambda}\nabla_j\, h_{ab} = 0, \tag{10.10)f}$$

and therefore,

$$K_{efg} = 0, \quad K_{abc} = 0. \tag{10.11}$$

The remaining steps of the proof follow a pattern similar to the one used in the proof of Theorem (8.1).

REMARK. The equations (10.10)a may be condensed into one set:

$$Q^{\omega\mu}\overset{xy}{P}{}_\lambda{}^\nu \overset{xy}{S}_{\omega\mu\nu} = 0,\ (xy = 12\, or\, xy = 34).$$

Contrary to the situation in the previous degenerate cases, condition (5.1) now imposes a restriction on $g_{\lambda\mu}$. The next two Theorems, which describe this situation, will be proved simultaneously.

THEOREM (10.5). *A necessary and sufficient condition that the solution described in Theorem* (10.4) *satisfy* (5.1) *is*

$$\overset{xy}{P}{}_\omega{}^\nu K_{\nu\lambda\mu} \overset{zt}{P}{}^{\lambda\mu} = 0 \tag{10.12)a}$$

$$xy = 12,\ 34; \quad xyzt \neq.$$

If this condition is satisfied, then

$$S_{afb} = 0,\ S_{afe} = 0,\ 2S_{abf} = K_{abf}, \tag{10.12)b}$$

$$(a \neq b,\ e \neq f),$$

while the remaining components of S remain the same as in Theorem (10.4).

THEOREM (10.6). *Condition* (10.12)a *for* $xy = 34$, $zt = 12$ *is equivalent to the requirement that the basic vectors* $\underset{1}{a}^{\nu}$ *and* $\underset{2}{a}^{\nu}$ *be* X_2-*forming.*

PROOF. We have for $a \neq b$ according to (10.10)f

$$K_{fab} = \nabla_f \, k_{ba} + \nabla_a \, k_{fb} + \nabla_b \, k_{fa} = -2\nabla_{(a} k_{b)f} = K_{f(ab)}, \tag{10.13)a}$$

and moreover,

$$K_{fab} = K_{fab} - (\underset{a}{\lambda}\nabla_a \, h_{fb} + \underset{b}{\lambda}\nabla_b \, h_{fa}) = = 2\underset{a}{\lambda}\{{}_{[b}{}^{e}{}_{a]}\}, \quad e \neq f. \tag{10.13)b}$$

Hence, the condition (10.12)a for $zt = 12$ is equivalent to

$$\{{}_{[a}{}^{e}{}_{b]}\} = 0, \tag{10.13)c}$$

so that $\underset{1}{a}^{\nu}$ and $\underset{2}{a}^{\nu}$ are X_2-forming. The condition (5.1) is equivalent to (8.4), *i.e.*,

$$S_{fab} + S_{fba} + S_{fef} = 0$$

$$S_{afe} + S_{aef} + S_{aba} = 0$$

$$(a \neq b, \ e \neq f),$$

or according to (10.10) and (10.13)a

$$\begin{aligned} 0 &= -\, S_{fef} = 2S_{f(ab)} = \tfrac{1}{2}K_{f(ab)} = \tfrac{1}{2}K_{fab} \\ 0 &= -\, S_{aba} = 2S_{a(fe)} = K_{a(fe)} = K_{afe}{}^{1} \\ &(a \neq b, \ e \neq f). \end{aligned} \tag{10.14}$$

The conditions (10.14) are equivalent to (10.12)a. Substituting from (10.14) into (10.10)c,d,e, we obtain (10.12)b.

NOTE. The singular case characterized by $\mathfrak{h} \overset{\text{def}}{=} \text{Det}\ ((g_{(\lambda\mu)}))=0$ is dealt with in Hlavatý, V., (21).

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[1] We have

$$K_{afe} = \nabla_a k_{ef} + \nabla_f k_{ae} + \nabla_e k_{af}$$

$$= -\, 2\nabla_{(f} k_{e)a} = K_{a(fe)}.$$

CHAPTER IV

GEOMETRY OF FIELD EQUATIONS

1. Curvature tensor. By *field equations* we mean a set of partial differential equations for $g_{\lambda\mu}$. This chapter concerns the *geometry* of field equations. Their *physical* application will be studied in the next chapter. Some of the results of this chapter are so formulated that they hold for any number n of dimensions. In all cases we assume

$$g(g-2) \neq 0.$$

This section is devoted to the study of the *curvature tensor.* Let $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ be an arbitrary connection; let

$$S_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=} \Gamma_{[\lambda}{}^{\nu}{}_{\mu]}, \quad S_{\lambda} \overset{\text{def}}{=} S_{\lambda\alpha}{}^{\alpha}$$

be its tensor and vector of torsion and D_{μ} the symbol of the covariant derivative with respect to $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$:

$$D_{\mu}v^{\nu} = \partial_{\mu}v^{\nu} + \Gamma_{\lambda}{}^{\nu}{}_{\mu}v^{\lambda} \tag{1.1)a}$$

$$D_{\mu}w_{\lambda} = \partial_{\mu}w_{\lambda} - \Gamma_{\lambda}{}^{\nu}{}_{\mu}w_{\nu}. \tag{1.1)b}$$

THEOREM (1.1). *Put*

$$R_{\omega\mu\lambda}{}^{\nu} \overset{\text{def}}{=} 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^{\nu}{}_{\omega]} + \Gamma_{\alpha}{}^{\nu}{}_{[\mu}\Gamma_{|\lambda|}{}^{\alpha}{}_{\omega]}). \tag{1.2}$$

Then $R_{\omega\mu\lambda}{}^{\nu}$ *is a tensor involved in the following identities*

$$2D_{[\omega}D_{\mu]}v^{\nu} = -R_{\omega\mu\lambda}{}^{\nu}v^{\lambda} + 2S_{\omega\mu}{}^{\alpha}D_{\alpha}v^{\nu} \tag{1.3)a}$$

$$2D_{[\omega}D_{\mu]}w_{\lambda} = R_{\omega\mu\lambda}{}^{\nu}w_{\nu} + 2S_{\omega\mu}{}^{\alpha}D_{\alpha}w_{\lambda}. \tag{1.3)b}$$

PROOF. We obtain from (1.1)

$$\begin{aligned}
2D_{[\omega}D_{\mu]}v^{\nu} &= 2[\partial_{[\omega}D_{\mu]}v^{\nu} - \Gamma_{[\mu}{}^{\alpha}{}_{\omega]}D_{\alpha}v^{\nu} + \Gamma_{\alpha}{}^{\nu}{}_{[\omega}D_{\mu]}v^{\alpha}] = \\
&= 2[\partial_{[\omega}\partial_{\mu]}v^{\nu} + v^{\lambda}\partial_{[\omega}\Gamma_{|\lambda|}{}^{\nu}{}_{\mu]} + \Gamma_{\lambda}{}^{\nu}{}_{[\mu}\partial_{\omega]}v^{\lambda} + \\
&\quad + \Gamma_{\alpha}{}^{\nu}{}_{[\omega}\partial_{\mu]}v^{\alpha} + \Gamma_{\alpha}{}^{\nu}{}_{[\omega}\Gamma_{|\lambda|}{}^{\alpha}{}_{\mu]}v^{\lambda} + S_{\omega\mu}{}^{\alpha}D_{\alpha}v^{\nu}] \\
&= -R_{\omega\mu\lambda}{}^{\nu}v^{\lambda} + 2S_{\omega\mu}{}^{\alpha}D_{\alpha}v^{\nu}.
\end{aligned}$$

The relation (1.3)b could be obtained in a similar way. The difference

$$D_{[\omega}D_{\mu]}v^{\nu} - S_{\omega\mu}{}^{\alpha}D_{\alpha}v^{\nu} = -\tfrac{1}{2}R_{\omega\mu\lambda}{}^{\nu}v^{\lambda}$$

is a tensor for any arbitrary vector v^ν. Hence, $R_{\omega\mu\lambda}{}^\nu$ must be a tensor.

DEFINITION (1.1). *The tensor* $R_{\omega\mu\lambda}{}^\nu$ *is termed the curvature tensor of* $\Gamma_\lambda{}^\nu{}_\mu$. *The tensors*

$$R_{\mu\lambda} \overset{\text{def}}{=} R_{\alpha\mu\lambda}{}^\alpha, \quad V_{\omega\mu} \overset{\text{def}}{=} R_{\omega\mu\alpha}{}^\alpha \tag{1.4}$$

are called the first and second contracted curvature tensors of $\Gamma_\lambda{}^\nu{}_\mu$ *respectively.*

NOTE I. The equations (1.3) can be generalized for arbitrary tensors:

$$\begin{aligned} 2D_{[\omega}D_{\mu]}T_{\lambda\cdots\eta}^{\nu\cdots\xi} = \\ = - R_{\omega\mu\alpha}{}^\nu T_{\lambda\cdots\eta}^{\alpha\cdots\xi} - \dots - R_{\omega\mu\alpha}{}^\xi T_{\lambda\cdots\eta}^{\nu\cdots\alpha} \\ + R_{\omega\mu\lambda}{}^\alpha T_{\alpha\cdots\eta}^{\nu\cdots\xi} + \dots + R_{\omega\mu\eta}{}^\alpha T_{\lambda\cdots\alpha}^{\nu\cdots\xi} \\ + 2S_{\omega\mu}{}^\alpha D_\alpha T_{\lambda\cdots\eta}^{\nu\cdots\xi}. \end{aligned} \tag{1.3c}$$

Moreover, if $\mathfrak{a}$ is a density of weight w, then

$$2D_{[\omega}D_{\mu]}\mathfrak{a} = wV_{\omega\mu}\mathfrak{a} + 2S_{\omega\mu}{}^\alpha D_\alpha\mathfrak{a}. \tag{1.3d}$$

NOTE II. For the algebraic decomposition of $R_{\omega\mu\lambda}{}^\nu$ see Lamson, K., (1).

THEOREM (1.2). *The curvature tensor* $R_{\omega\mu\lambda}{}^\nu$ *satisfies the following identities:*

$$R_{\omega\mu\lambda}{}^\nu = R_{[\omega\mu]\lambda}{}^\nu \tag{1.5a}$$

$$R_{[\omega\mu\lambda]}{}^\nu = 4S_{[\omega\mu}{}^\alpha S_{\lambda]\alpha}{}^\nu + 2D_{[\omega}S_{\mu\lambda]}{}^\nu \tag{1.5b}$$

$$D_{[\xi}R_{\omega\mu]\lambda}{}^\nu = - 2S_{[\xi\omega}{}^\alpha R_{\mu]\alpha\lambda}{}^\nu \tag{1.5c}$$

(*Bianchi's identity*).

PROOF. (1.5)a follows immediately from (1.2). In order to prove (1.5)b,c we use an obvious relation

$$2D_{[\xi}(D_\omega D_{\mu]}v^\nu) = 2D_{[\xi}D_\omega(D_{\mu]}v^\nu) \tag{1.5d}$$

valid for any vector v^ν. From (1.3)c we obtain

$$\begin{aligned} 2D_{[\xi}D_\omega(D_{\mu]}v^\nu) = R_{[\xi\omega\mu]}{}^\alpha D_\alpha v^\nu \\ - (D_{[\mu}v^\lambda)R_{\xi\omega]\lambda}{}^\nu + 2(D_\alpha D_{[\mu}v^\nu)S_{\xi\omega]}{}^\alpha, \end{aligned}$$

so that the right-hand term in (1.5)d is

$$\begin{aligned} 2D_{[\xi}D_{\omega}(D_{\mu]}v^{\nu}) = (R_{[\xi\omega\mu]}{}^{\beta} &- 4S_{[\xi\omega}{}^{\alpha}S_{\mu]\alpha}{}^{\beta})D_{\beta}v^{\nu} \\ &+ 2S_{[\xi\omega}{}^{\alpha}R_{\mu]\alpha\lambda}{}^{\nu}v^{\lambda} \\ + 2S_{[\xi\omega}{}^{\alpha}D_{\mu]}D_{\alpha}v^{\nu} &- (D_{[\mu}v^{\lambda})R_{\xi\omega]\lambda}{}^{\nu}. \end{aligned}$$

The left-hand term of (1.5)d may be computed in a similar manner. Substituting these results into (1.5)d and equating the coefficients of v^{λ} and $D_{\beta}v^{\nu}$, we obtain (1.5)b,c.

THEOREM (1.3).[1] *If* II (2.1) *holds, then*

(1.6)a $$V^{\alpha\beta}_{\xi\omega\mu\lambda}\,g_{\alpha\beta} = 0$$

where

(1.6)b $$V^{\alpha\beta}_{\xi\omega\mu\lambda} \overset{\text{def}}{=} \tfrac{1}{2}\delta^{\beta}_{\mu}R_{\xi\omega\lambda}{}^{\alpha} + \\ + \delta^{\alpha}_{\lambda}[\tfrac{1}{2}R_{\xi\omega\mu}{}^{\beta} + 2S_{\xi\omega}{}^{\gamma}S_{\gamma\mu}{}^{\beta} - 2D_{[\xi}S_{\omega]\mu}{}^{\beta} + 4S_{\mu[\omega}{}^{\gamma}S_{\xi]\gamma}{}^{\beta}].$$

PROOF. From II (2.1)b we have

$$2D_{[\xi}D_{\omega]}g_{\lambda\mu} = 4D_{[\xi}(S_{\omega]\mu}{}^{\alpha}g_{\lambda\alpha}).$$

Substituting into the left hand term from

$$2D_{[\xi}D_{\omega]}g_{\lambda\mu} = R_{\xi\omega\lambda}{}^{\alpha}g_{\alpha\mu} + R_{\xi\omega\mu}{}^{\alpha}g_{\lambda\alpha} + 2S_{\xi\omega}{}^{\alpha}D_{\alpha}g_{\lambda\mu}$$

and using II (2.1)b again, one gets (1.6).

THEOREM (1.4). *The contracted curvature tensors are related by*

(1.7) $$2R_{[\omega\mu]} + V_{\omega\mu} = 2(D_{\alpha}S_{\omega\mu}{}^{\alpha} + 2\partial_{[\omega}S_{\mu]}).$$

If II (2.1)b *holds, then*

(1.8)a $$R_{[\omega\mu]} = D_{\alpha}S_{\omega\mu}{}^{\alpha} + \partial_{[\omega}S_{\mu]}$$

(1.8)b $$V_{\omega\mu} = 2\partial_{[\omega}S_{\mu]}.$$

PROOF. Summing for $\omega = \nu$ in (1.5)b we obtain (1.7). If II (2.1)b holds, so does II (2.6)a, and we get by virtue of (1.3)d

$$\begin{aligned} 2D_{[\xi}D_{\omega]}\mathfrak{g} &= 2V_{\xi\omega}\mathfrak{g} + 4S_{\xi\omega}{}^{\alpha}S_{\alpha}\mathfrak{g} = \\ &= 4(D_{[\xi}S_{\omega]})\mathfrak{g} + 4S_{[\omega}D_{\xi]}\mathfrak{g} = \\ &= 4(D_{[\xi}S_{\omega]})\mathfrak{g} \end{aligned}$$

---

[1] *Cf.* also Bose, S. N., (3).

or

$$V_{\xi\omega} = 2(D_{[\xi}S_{\omega]} - S_{\xi\omega}{}^{\alpha}S_{\alpha}) = 2\partial_{[\xi}S_{\omega]}.$$

Substituting from (1.8)b into (1.7) one obtains (1.8)a.

2. Field equations. From now on we shall assume that II (2.1) holds

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha}g_{\lambda\alpha}. \tag{2.1}$$

THEOREM (2.1). *Put*

$$P_{\lambda\mu} \overset{\text{def}}{=} \partial_\alpha \Gamma_\lambda{}^\alpha{}_\mu - \Gamma_\lambda{}^\alpha{}_\beta \Gamma_\alpha{}^\beta{}_\mu + \Gamma_\lambda{}^\alpha{}_\mu \Gamma_{(\alpha}{}^\beta{}_{\beta)} - \tfrac{1}{2}(\partial_\mu \Gamma_{(\lambda}{}^\alpha{}_{\alpha)} + \partial_\lambda \Gamma_{(\mu}{}^\alpha{}_{\alpha)}). \tag{2.2)a}$$

Then

$$R_{\mu\lambda} + P_{\lambda\mu} = D_\mu S_\lambda \tag{2.2)b}$$

so that $P_{\lambda\mu}$ is a tensor.

PROOF. An easy inspection shows that II (2.6)a is equivalent to

$$\Gamma_\alpha{}^\alpha{}_\omega = \tfrac{1}{2}\partial_\omega \ln \mathfrak{g} - S_\omega. \tag{2.3)a}$$

Adding $2S_\omega$ to both sides of this equation, one obtains

$$\Gamma_\omega{}^\alpha{}_\alpha = \tfrac{1}{2}\partial_\omega \ln \mathfrak{g} + S_\omega \tag{2.3)b}$$

so that

$$2\Gamma_{(\omega}{}^\alpha{}_{\alpha)} = \partial_\omega \ln \mathfrak{g}.\,^{1} \tag{2.3)c}$$

On the other hand, we see from (1.4) and (1.2) that

$$R_{\mu\lambda} = \partial_\mu \Gamma_\lambda{}^\alpha{}_\alpha - \partial_\alpha \Gamma_\lambda{}^\alpha{}_\mu + \Gamma_\alpha{}^\beta{}_\mu \Gamma_\lambda{}^\alpha{}_\beta - \Gamma_\alpha{}^\beta{}_\beta \Gamma_\lambda{}^\alpha{}_\mu. \tag{2.4}$$

The equation (2.2)b follows easily from (2.2)a, (2.3), and (2.4).

Einstein, A., (6) first proposed the following set of twenty field equations

$$S_\lambda = 0 \tag{2.5)a}$$

$$P_{\lambda\mu} = 0 \tag{2.5)b}$$

for sixteen unknowns $g_{\lambda\mu}$. From Theorem (2.1) we see that the second set reduces by virtue of the first set to

$$R_{\lambda\mu} = 0.\,^{2,\,3} \tag{2.5)c}$$

---

[1] This formula follows also from II (7.1)c.

[2] Einstein had a formula equivalent to (2.3)c. He does not have the formulae (2.3)a,b.

[3] *Cf.* also Einstein, A. — Kaufman, B., (1).

Later[1] Einstein proposed a weaker set consisting of eighteen field equations (2.5)a and

$$\text{(2.6)} \qquad \text{a) } R_{(\mu\lambda)} = 0, \quad \text{b) } \partial_{[\omega} R_{\mu\lambda]} = 0.$$

The set (2.6)b is obviously equivalent to

$$\text{(2.6)c} \qquad R_{[\mu\lambda]} = \partial_{[\mu} X_{\lambda]}$$

where X_λ is an arbitrary covariant vector. The set consisting of (2.6)a,c also includes (2.5)c. One obtains it by choosing a gradient for X_λ.

Einstein's unified theory is governed by the following set of equations: sixty-four equations determining the connection $\Gamma_\lambda{}^\nu{}_\mu$,

$$\text{(2.7)} \qquad D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}$$

and twenty field equations

$$\text{(2.8)a} \qquad S_\lambda = 0$$

$$\text{(2.8)b} \qquad R_{[\mu\lambda]} = \partial_{[\mu} X_{\lambda]}$$

$$\text{(2.8)c} \qquad R_{(\mu\lambda)} = 0$$

for twenty unknowns $g_{\lambda\mu}$ and X_λ. Substituting the solution $\Gamma_\lambda{}^\nu{}_\mu$ from (2.7) into (2.8) one obtains four differential equations (2.8)a of the first order and sixteen differential equations of the second order for $g_{\lambda\mu}$. Hence, there are twenty differential equations (2.8) for twenty unknowns, $g_{\lambda\mu}$ and X_λ, and sixty-four algebraic equations (2.7) for sixty-four unknowns, $\Gamma_\lambda{}^\nu{}_\mu$.[2]

NOTE. The systems (2.7) and (2.8) may be considered from another point of view.

One could start with sixteen differential equations (2.8)b,c of the first order with respect to the unknowns $\Gamma_\lambda{}^\nu{}_\mu$ which unknowns are subject to the algebraic system (2.8)a. Then the system (2.7) has to be considered as an overdetermined system of sixty-four differential equations of the first order with respect to the unknowns $g_{\lambda\mu}$. In this way the system consisting of (2.7) and (2.8) is split into two systems each of the first order and an algebraic system. We shall see later that this way of considering the fundamental equations (2.7) and (2.8) permits us to solve them by elementary *algebraic* means.

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[1] *Cf.* Einstein, A. — Kaufman, B., (1).

[2] Stephenson, G., (2) replaces (2.8)a by another condition which amounts to $R_{\omega\mu\alpha}{}^\alpha = k_{\omega\mu}$.

3. Continuation. The curvature tensor $H_{\omega\mu\lambda}{}^{\nu}$ of $\{{}_{\lambda}{}^{\nu}{}_{\omega}\}$ is defined in the same way as is the curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ of $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$:

$$H_{\omega\mu\lambda}{}^{\nu} \overset{\text{def}}{=} 2\partial_{[\mu}\{{}_{|\lambda|}{}^{\nu}{}_{\omega]}\} + 2\{{}_{\alpha}{}^{\nu}{}_{[\mu}\}\{{}_{|\lambda|}{}^{\alpha}{}_{\omega]}\}. \tag{3.1}$$

All results of the first section are valid for

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\}. \tag{3.2}$$

The most important of these results are the following: The identities (1.3)a,b,c,d reduce to

$$2\nabla_{[\omega}\nabla_{\mu]}v^{\nu} = -H_{\omega\mu\lambda}{}^{\nu}v^{\lambda} \tag{3.3)a (}$$

$$2\nabla_{[\omega}\nabla_{\mu]}w_{\lambda} = H_{\omega\mu\lambda}{}^{\nu}w_{\nu} \tag{3.3)b (}$$

$$\begin{aligned}2\nabla_{[\omega}\nabla_{\mu]}T^{\nu\cdots\xi}_{\lambda\cdots\eta} = {} & -H_{\omega\mu\alpha}{}^{\nu}T^{\alpha\cdots\xi}_{\lambda\cdots\eta} - \ldots - H_{\omega\mu\alpha}{}^{\xi}T^{\nu\cdots\alpha}_{\lambda\cdots\eta} + \\ & + H_{\omega\mu\lambda}{}^{\alpha}T^{\nu\cdots\xi}_{\alpha\cdots\eta} + \ldots + H_{\omega\mu\eta}{}^{\alpha}T^{\nu\cdots\xi}_{\lambda\cdots\alpha}\end{aligned} \tag{3.3)c (}$$

$$2\nabla_{[\omega}\nabla_{\mu]}\mathfrak{a} = 0. \tag{3.3)d (}$$ [1]

The identities (1.5) reduce to

$$H_{\omega\mu\lambda}{}^{\nu} = H_{[\omega\mu]\lambda}{}^{\nu} \tag{3.4)a (}$$

$$H_{[\omega\mu\lambda]}{}^{\nu} = 0 \tag{3.4)b (}$$

$$\nabla_{[\xi}H_{\omega\mu]\lambda}{}^{\nu} = 0 \text{ (Bianchi's identity).} \tag{3.4)c (}$$

The equation (2.1) has to be replaced here by

$$\nabla_{\omega}h_{\lambda\mu} = 0, \tag{3.5}$$

and consequently, (1.6)a reduces to

$$H_{\omega\mu(\lambda\nu)} = 0. \tag{3.4)d (}$$

By the same token we have from (1.8)b

$$H_{\omega\mu\alpha}{}^{\alpha} = 0 \tag{3.6}$$

while, according to (1.7), the tensor

$$H_{\mu\lambda} \overset{\text{def}}{=} H_{\alpha\mu\lambda}{}^{\alpha} \tag{3.7)a (}$$

is symmetric

$$H_{[\mu\lambda]} = 0. \tag{3.7)b (}$$

Therefore, an easy inspection leads us from (3.4)c to

$$\nabla_{\lambda}(H^{\lambda\nu} - \tfrac{1}{2}Hh^{\lambda\nu}) = 0 \tag{3.4)e (}$$

---

[1] We are anticipating here the result described by (3.6).

where

$$H \overset{\text{def}}{=} H_{\lambda\mu} h^{\lambda\mu}. \tag{3.8}$$

NOTE. Equation (3.4)e is a consequence of Bianchi's identity (3.4)c. If one uses the generalized Bianchi identity (1.5)c, one could express

$$D_{\lambda}(R_{\mu}{}^{\lambda} - \tfrac{1}{2}R\delta_{\mu}^{\lambda})$$ [1]

in terms of the curvature tensor and the torsion tensor. In this way one obtains a generalization of (3.4)e. [2] [3]

The field equations (2.8)b,c exhibit some properties of $R_{\mu\lambda}$. We can express them by means of $H_{\mu\lambda}$. [4] To this effect we shall first prove the following

THEOREM (3.1). *Put*

$$T_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=} S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}. \tag{3.9}$$

Then

$$R_{\omega\mu\lambda}{}^{\nu} = H_{\omega\mu\lambda}{}^{\nu} + 2\nabla_{[\mu} T_{|\lambda|\omega]}{}^{\nu} + 2T_{\alpha[\mu}{}^{\nu} T_{|\lambda|\omega]}{}^{\alpha}. \tag{3.10}$$

PROOF. We have

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} + T_{\lambda\mu}{}^{\nu}. \tag{3.11}$$

Substituting (3.11) into (1.2) and remembering (3.1), we obtain (3.10).

In the next Theorem we shall use the following abbreviations

(3.12)a
$$\begin{aligned} {}^{*}A_{[\omega\mu]} \overset{\text{def}}{=} {} & 2\nabla^{\alpha} U^{\beta}{}_{\alpha[\omega} k_{\mu]\beta} \\ & + 2U^{\beta}{}_{\alpha[\omega}(\tfrac{1}{2}K_{\mu]\beta}{}^{\alpha} - k_{\mu]}{}^{\gamma} U_{\gamma}{}^{\alpha}{}_{\beta} + U^{\gamma\alpha}{}_{\mu]} k_{\beta\gamma}) + \\ & + U^{\beta}(\tfrac{1}{2}K_{\omega\mu\beta} - 2U_{\alpha\beta[\mu} k_{\omega]}{}^{\alpha}), \end{aligned}$$

(3.12)b
$$\begin{aligned} -{}^{*}A_{(\mu\lambda)} \overset{\text{def}}{=} {} & -\nabla_{\alpha} U^{\alpha}{}_{\lambda\mu} + U^{\beta}{}_{\alpha\mu} U^{\alpha}{}_{\beta\lambda} + \nabla_{\mu} U_{\lambda} \\ & - U_{\beta} U^{\beta}{}_{\lambda\mu} - (K_{\lambda\beta}{}^{\alpha} U_{\gamma}{}^{\beta}{}_{[\mu} + K_{\mu\beta}{}^{\alpha} U_{\gamma}{}^{\beta}{}_{[\lambda}) k_{\alpha]}{}^{\gamma} \\ & + 4U_{\gamma}{}^{\beta}{}_{[\mu} k_{\alpha]}{}^{\gamma} U_{\delta}{}^{\alpha}{}_{[\beta} k_{\lambda]}{}^{\delta}. \end{aligned}$$ [5]

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[1] Here we put

$$R_{\mu}{}^{\lambda} \overset{\text{def}}{=} R_{\mu\nu} h^{\nu\lambda}, \qquad R \overset{\text{def}}{=} R_{\mu\lambda} h^{\mu\lambda}.$$

[2] The computation requires the use of III (3.5)c.

[3] *Cf.* also Bose, S. N., (1), Einstein, A., (3) and Lichnerowicz, A., (3), (4); Schrödinger, E., (4); Winogradski, J., (5).

[4] This result will be necessary for physical applications.

[5] Tensor $K_{\omega\mu\nu}$ is defined by II (3.5). Vector U_{λ} is defined by II (7.1)c.

THEOREM (3.2). *The field equations* (2.8)b,c *are equivalent to*

$$\begin{aligned}\partial_{[\omega} X_{\mu]} &= \tfrac{1}{2}\nabla^{\alpha} K_{\omega\mu\alpha} + {}^{*}A_{[\omega\mu]} \\ &\quad + \partial_{[\omega} S_{\mu]} + S^{\beta}(\tfrac{1}{2}K_{\omega\mu\beta} - 2U_{\alpha\beta[\mu}k_{\omega]}{}^{\alpha})\end{aligned} \tag*{(3.13)a}$$

$$H_{\mu\lambda} - \tfrac{1}{4}K_{\alpha\mu}{}^{\beta}K_{\beta\lambda}{}^{\alpha} - {}^{*}A_{(\mu\lambda)} + \nabla_{(\mu} S_{\lambda)} - S_{\beta}U^{\beta}{}_{\lambda\mu} = 0. \tag*{(3.13)b}$$

PROOF. The equation (3.10) leads to

$$\begin{aligned}R_{\mu\lambda} &= H_{\mu\lambda} + \nabla_{\mu} T_{\lambda} - \nabla_{\alpha} T_{\lambda\mu}{}^{\alpha} \\ &\quad + T_{\alpha\mu}{}^{\beta} T_{\lambda\beta}{}^{\alpha} - T_{\alpha} T_{\lambda\mu}{}^{\alpha}\end{aligned} \tag*{(3.14)a}$$

where

$$T_{\lambda} \overset{\text{def}}{=} T_{\lambda\alpha}{}^{\alpha}. \tag*{(3.14)b}$$

From (3.9) we obtain by virtue of (3.14)b and II (3.6)

$$T_{\lambda} = S_{\lambda} + U_{\lambda} \tag*{(3.15)a}$$

$$T_{\omega\mu}{}^{\nu} = \tfrac{1}{2}K_{\omega\mu}{}^{\nu} - 2U^{\alpha\nu}{}_{[\mu}k_{\omega]\alpha} + U^{\nu}{}_{\omega\mu}. \tag*{(3.15)b}$$

Substituting from (3.15)a,b into (3.14)a and remembering (3.7)b and II (7.1)c, one obtains (3.13) by virtue of (2.8)b,c.

NOTE I. Using II (7.23) one could express T_{λ} in terms of $U_{\lambda\nu\mu}$. This, however, is not necessary because of (2.8)a.

NOTE II. It is illuminating to compare the fundamental equations of the gravitational theory with those of the unified theory. The fundamental equations of the former theory consist of (3.5) and

$$H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = {}^{*}k\,{}^{*}T_{\lambda\mu}, \tag*{(3.16)}$$

where ${}^{*}k$ is a constant and ${}^{*}T_{\lambda\mu}$ is the momentum energy tensor of matter. The fundamental equations of the unified theory consist of the sets (2.7), (2.8)a and (3.13). Hence, physics is not involved in the formulation of these latter equations. There is still another difference between the two systems of the fundamental equations. In trying to solve the fundamental equations of the gravitational theory one starts with the assumption that the matrix $((h_{\lambda\mu}))$ is a diagonal one. This assumption is at least partly justified by the observable spatial isotropy of our three dimensional space. The Schwarzschild diagonal form of $((h_{\lambda\mu}))$ is enforced by Lamé's conditions which have to be incorporated into the fundamental equations (3.5) and (3.16).

Their presence in this connection is at least partly justified by the spatial isotropy of our three-dimensional space. Moreover, the restriction imposed by them on the solution $h_{\lambda\mu}$ is at least partly counterbalanced by $^*T_{\lambda\mu}$ whose form has not been exactly described.

This situation changes considerably for the unified theory where we are faced with the problem considered in section II (21). We see from II (21.9) that $((h_{\lambda\mu}))$ is again a diagonal matrix, and this restriction could be justified as before. However, there are more restrictions dealing with $k_{\lambda\mu}$ which are not justified by observable data. They require new conditions for $g_{\lambda\mu}$, namely II (21.8)a, *i.e.*,

$$\underset{x}{E}_{[\omega}{}^{\alpha}\partial_{\mu}\underset{x}{E}_{\lambda]}{}^{\beta} = 0,$$

which have to be incorporated in the system of fundamental equations. We know that this set is equivalent to a set of sixteen conditions. These new conditions are not justified by observable data and, what is more unfortunate, they are not counterbalanced by a more or less free choice of a tensor involved in the theory.[1] Therefore, it is doubtful whether solutions $g_{\lambda\mu}$ obtained from this enlarged system of thirty-six field equations are suitable for physical applications. Thus, for instance, the third class, very important for celestial mechanics, is not included among these solutions. Of course, a similar criticism could be directed against the conditions II (6.4), II (7.4)a and II (7.22)a. However, the latter two conditions may be easily converted into physical requirements as we shall see later.

4. Uniqueness Theorem. In this section we shall investigate the relationship between the field equations (2.8) and the tensor field defined by I (15.2), *i.e.*,

$$f_{\lambda\nu} \overset{\text{def}}{=} \frac{\sigma}{\sqrt{g}} [\kappa\sqrt{|k|}k_{\lambda\nu} - \tfrac{1}{2}e_{\lambda\nu\alpha\beta}k^{\alpha\beta}]. \tag{4.1}$$

THEOREM (4.1). *The conditions* (2.8)a *and*

$$\partial_{[\omega}f_{\mu\lambda]} = 0 \tag{4.2a}$$

are equivalent.

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[1] We have twenty equations (2.8) for twenty unknowns $g_{\lambda\mu}$, X_{λ}.

PROOF. The equation I (15.1) yields

$$\partial_{[\omega} f_{\mu\lambda]} = -\frac{\sigma}{2}\, \mathfrak{e}_{\alpha\beta[\mu\lambda}\partial_{\omega]}({}^*k^{\alpha\beta}\sqrt{|\mathfrak{g}|})$$

equivalent to

$$3\partial_{[\omega} f_{\mu\lambda]} = -\,\sigma\mathfrak{e}_{\omega\mu\lambda\nu}\partial_\alpha({}^*k^{\alpha\nu}\sqrt{|\mathfrak{g}|}). \tag{4.3}$$

Therefore, we obtain by virtue of II (6.14)

$$3\sigma\partial_{[\omega} f_{\mu\lambda]} = S_\alpha\sqrt{|\mathfrak{g}|}\ {}^*h^{\alpha\nu}\mathfrak{e}_{\omega\mu\lambda\nu}. \tag{4.4}$$

Our statement is then an immediate consequence of (4.4) and Theorem I (14.1).

NOTE I. If and only if (4.2)a holds, then there is a vector f_λ such that

$$f_{\lambda\mu} = \partial_{[\lambda} f_{\mu]}. \tag{4.2)b}$$

Therefore, we see from I (15.17) that the field equation (2.8)a enables us *to express sixteen components* $g_{\lambda\mu}$ *in terms of fourteen components* $h_{\lambda\mu}$ *and* f_λ,

$$g_{\lambda\nu} = h_{\lambda\nu} + \frac{\sigma}{\sqrt{e}}\,[\tfrac{1}{2}e_{\lambda\nu\alpha\beta}\nabla^\alpha f^\beta - \varphi\sqrt{|f|}\,\partial_{[\lambda} f_{\;]}]. \tag{4.5}$$

NOTE II. The question arises whether there is another skew-symmetric tensor $m_{\lambda\nu}$ besides $f_{\lambda\nu}$ for which we have

$$\partial_{[\omega} m_{\mu\lambda]} = 0$$

by virtue of the field equations. To answer this question we first need a definition and two Lemmas.

DEFINITION (4.1). *A tensor field* $m_{\lambda\mu}$ *is termed a Maxwell tensor field belonging to* $g_{\lambda\mu}$ *if it satisfies the following conditions:*

a) *it is skew-symmetric,*
b) *it is obtainable from* $g_{\lambda\mu}$ *by a purely tensorial construction,*[1]
c) *it is a function of* $g_{\lambda\mu}$ *alone and does not contain derivatives of* $g_{\lambda\mu}$,
d) *it satisfies the condition*

$$\partial_{[\omega} m_{\mu\lambda]} = 0 \tag{4.6}$$

if and only if the field equations (2.8) *are satisfied.*

---

[1] *i.e.*, by a construction every step of which is the same in every coordinate system and the resulting tensor is defined up to a constant factor of proportionality.

The following Lemma is concerned with the tensor $\overset{xy}{Q}_{\omega\mu}$ defined by I (10.6)b (see also I (10.8). Hence, this Lemma involves only the first two classes.

LEMMA (4.1). *The tensors* $\overset{xy}{Q}_{\omega\mu}$ *may be expressed as linear combinations of* $k_{\lambda\mu}$ *and* $f_{\lambda\mu}$

$$\overset{xy}{Q}_{\omega\mu} = \overset{xy}{a}\, k_{\omega\mu} + \overset{xy}{b} f_{\omega\mu} \tag{4.7)a}$$

where the scalars a, b satisfy the condition

$$\begin{vmatrix} \overset{12}{a} & \overset{12}{b} \\ \overset{34}{a} & \overset{34}{b} \end{vmatrix} \neq 0. \tag{4.7)b}$$

PROOF. If we substitute from (4.1) for $e_{\lambda\nu\alpha\beta}k^{\alpha\beta}$ into I (10.6)b we obtain (4.7)a where $\overset{xy}{a}$ and $\overset{xy}{b}$ are obviously scalar functions of $g_{\lambda\mu}$. From I (10.8) we see that $\overset{12}{Q}_{\omega\mu}$ and $\overset{34}{Q}_{\omega\mu}$ are not proportional. Therefore, (4.7)b holds.

In the next Lemma we shall again confine ourselves to the first two classes.

LEMMA (4.2). *Every tensor field* $m_{\lambda\mu}$ *satisfying the conditions* a) b) c) *of Definition* (4.1) *is of the form*

$$m_{\lambda\mu} = x f_{\lambda\mu} + y k_{\lambda\mu} \tag{4.8}$$

where x, y are scalar functions of $g_{\lambda\mu}$ *which contain no derivatives of* $g_{\lambda\mu}$.

PROOF. We know from I (8.1)a and I (9.3) that

$$g_\lambda{}^\nu \overset{\text{def}}{=} g_{\lambda\mu}h^{\mu\nu} = \sum_x (1 + \overset{x}{\lambda}) \underset{x}{a}_\lambda \underset{x}{a}^\nu. \tag{4.9}$$

On the other hand, any skew symmetric tensor $m_{\lambda\mu}$ may be expressed as follows:

$$m_{\lambda\mu} = \sum_{xy} m_{xy} \overset{x}{a}_{[\lambda} \overset{y}{a}_{\mu]}. \tag{4.10}$$

The construction of such a tensor requires the construction of the tensors $\overset{x}{a}_{[\lambda}\overset{y}{a}_{\mu]}$. From (4.9) we see that a purely tensorial construction based on $g_{\lambda\mu}$ and not on its derivatives could only provide us with tensors $\overset{x}{a}_\lambda \underset{x}{a}^\nu$. We achieved this result in finding the tensors $\underset{x}{E}_\lambda{}^\nu$ (see I (10.10) and I (10.11)). Hence, a purely tensorial construction of a quadratic tensor, based on $g_{\lambda\mu}$ and not on its derivatives, yields only tensors of the form

$$m_\lambda{}^\nu = \sum\nolimits^x m_x \underset{x}{E}_\lambda{}^\nu.$$

If $m_{\lambda\nu}$ is skew symmetric, we have by virtue of (4.10), I (10.11) and I (8.16)

$$m_1 = -m_2, \quad m_3 = -m_4$$

so that $m_{\lambda\mu}$ must have the form

$$m_{\lambda\mu} = \underset{12}{\alpha}\overset{12}{Q}_{\lambda\mu} + \underset{34}{\alpha}\overset{34}{Q}_{\lambda\mu}.$$

Substituting into this equation from (4.7)a we obtain (4.8). According to the manner in which we obtained this equation it is clear that x, y must be scalar functions of $g_{\lambda\mu}$ which contain no derivatives of $g_{\lambda\mu}$.

THEOREM (4.2). *In the case of the first class there is only one Maxwell tensor field (but for a multiplicative constant) belonging to* $g_{\lambda\mu}$, *namely the tensor field* $f_{\lambda\mu}$.

PROOF. According to Theorem (4.1) the tensor $f_{\lambda\mu}$ defined by (4.1) is obviously a Maxwell tensor field. According to Lemma (4.2) any other Maxwell tensor field $m_{\lambda\mu}$ must be of the form (4.8). Such a tensor is a Maxwell tensor field if and only if the condition d) of Definition (4.1) is satisfied. This condition is according to (4.2)a and (4.8) equivalent to

$$\partial_{[\omega} y k_{\mu\lambda]} + f_{[\mu\lambda}\partial_{\omega]} x = 0. \tag{4.11}$$

Using I (15.7) in the simplified form

$$k_{\mu\lambda} = \alpha f_{\mu\lambda} + \beta\sqrt{|\mathfrak{h}|}\,e_{\alpha\beta\mu\lambda} f^{\alpha\beta} \tag{4.12)a}$$

with

$$\alpha \overset{\text{def}}{=} -\frac{\sigma\varphi}{\sqrt{e}}\sqrt{|f|}, \quad \beta \overset{\text{def}}{=} \frac{\sigma}{2\sqrt{e}},^{1} \tag{4.12)b}$$

---

[1] See I (15.16) and I (2.11)b.

we may write, instead of (4.11),

$$(4.13) \qquad f_{[\mu\lambda}\partial_{\omega]}(x + \alpha y) + \mathfrak{e}_{\alpha\beta[\mu\lambda}\partial_{\omega]}(y\beta f^{\alpha\beta}\sqrt{|\mathfrak{h}|}).$$

This equation is equivalent to

$$(4.14) \qquad f_{[\mu\lambda}\partial_{\omega]}(x + y\alpha) + \tfrac{2}{3}\mathfrak{e}_{\omega\mu\lambda\nu}\partial_{\alpha}(y\beta f^{\alpha\nu}\sqrt{|\mathfrak{h}|}) = 0.$$

Assume first that (4.14) is not an identity. If $m_{\mu\lambda}$ is to be a Maxwell tensor field, (4.14) has to be satisfied by virtue of (2.8). Among this latter set of differential equations there is only one set of the first order, namely (2.8)a. Moreover, an easy inspection shows that the left-hand term of (4.14) cannot be a first integral of (2.8)b,c. Hence, (4.14) must be a consequence of (2.8)a alone. On the other hand, (2.8)a is equivalent to (4.2)a, and this latter equation is different from (4.14). Hence, (4.14) is not a consequence of (2.8)a, either. *Therefore, it represents a set of new conditions besides* (2.8)a. In the case of the first class there is only one set of conditions, namely (2.8). Hence, our result is at variance with the assumption that $g_{\lambda\mu}$ belongs to the first class. Therefore, (4.14) must be an identity

$$(4.15) \qquad x + y\alpha = c \;\; (= \text{const}),^{1} \;\; y\beta = 0.$$

Comparing the second equation (4.15) with the second equation (4.12)b, we obtain $y = 0$ so that the first equation (4.15) yields $x = c$. Substituting these values into (4.8) we have

$$m_{\lambda\mu} = cf_{\lambda\mu}.$$

NOTE I. Theorem (4.2) will help us in identifying $f_{\mu\lambda}$ with the electromagnetic field tensor.

NOTE II. If we replace (2.8)a by another condition, we obtain another Maxwell tensor field. Thus, for instance,

$$S_{[\omega\mu\lambda]} = 0$$

leads by virtue of II (6.12) to

$$\partial_{[\omega}k_{\mu\lambda]} = 0.$$

NOTE III. If we admit other restrictions besides (2.8), we obtain more than the one Maxwell tensor field, $f_{\lambda\mu}$. Thus, the condition

$$U^{\nu}{}_{\lambda\mu} = 2p^{\nu}h_{\lambda\mu} - \delta^{\nu}_{\lambda}p_{\mu} - \delta^{\nu}_{\mu}p_{\lambda}\,^{2}$$

---

[1] For the first class, $f_{\mu\nu}$ is not a simple bivector.

[2] See Theorems II (7.5) and II (7.6). Physical consequences and applications of this condition are described in Hlavatý, V., (18). *Cf.* also Clauser, E., (2).

yields a Maxwell tensor field

$$m_{\lambda\mu} = -\tfrac{1}{2}\sqrt{|\mathfrak{g}|}\, e_{\lambda\mu\alpha\beta} k^{\alpha\beta}. \tag{4.16}$$ [1]

NOTE IV. If we drop condition c) of Definition (4.1) we are able to find at least two fields satisfying (4.6). One is $f_{\mu\lambda}$ and the other is $R_{[\mu\lambda]}$ (*Cf.* (2.8)b). [2]

5. Digression: Solutions of (2.7) and (2.8). In this section we shall display a solution of (2.7) and (2.8). Assume $h_{\lambda\mu}$ to be of the form

$$((h_{\lambda\mu})) \overset{\text{def}}{=} \left(\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right), \tag{5.1}$$

introduce the abbreviation $z \overset{\text{def}}{=} x^{\mathrm{III}} - x^{\mathrm{IV}}$, denote by $\alpha = \alpha(z)$ an arbitrary function of z, possessing at least the first two derivatives with respect to z [3], and introduce the vectors

$$a_\lambda(0,\ 0,\ 1,\ -1) \tag{5.2)a}$$

$$u_\lambda(-\cos\alpha,\ \sin\alpha,\ 0,\ 0) \tag{5.2)b}$$

which satisfy the conditions

$$a^\lambda a_\lambda = u^\lambda a_\lambda = u^\lambda u_\lambda - 1 = 0. \tag{5.2)c}$$

The tensor $g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$ with

$$k_{\lambda\mu} = 2a_{[\lambda}u_{\mu]} \tag{5.3}$$

is obviously of the third class:

$$\mathfrak{k} = 0, \quad K = 0, \quad \mathfrak{g} = \mathfrak{h}. \tag{5.4}$$

---

[1] See preceding footnote.

[2] Finzi, B., (2) uses a tensor $^*R_{\mu\lambda}$ similar to $P_{\lambda\mu}$ (see (2.2.)b) and neglects (2.8)a. From his field equations

$$\operatorname{Rot} {}^*R_{\mu\lambda} = 0 \qquad \operatorname{Div} {}^*R^{\mu\lambda} = 0$$

it is clear that $^*R_{\mu\lambda}$ could be identified with the electromagnetic tensor field. *Cf.* also Zanella, A., (1); Martuscelli, L., (1); Schrödinger, E., (2); Graiff, F., (2).

[3] Derivatives with respect to z will be denoted by primes.

Moreover, since $z = x^{\mathrm{III}} - x^{\mathrm{IV}}$,

$$\text{(5.5)a} \qquad \nabla_{[\omega}\, k_{\mu\lambda]} = \partial_{[\omega} k_{\mu\lambda]} = 2a_{[\mu}\partial_{\omega} u_{\lambda]} = 2a_{[\mu} a_{\omega} u'_{\lambda]} = 0$$

$$\text{(5.5)b} \qquad \nabla_{\beta}\, k_{\lambda}{}^{\beta} = \partial_{\beta}(a_{\lambda} u^{\beta} - u_{\lambda} a^{\beta}) = 0,$$

so that by virtue of II (3.5) and (5.2)c

$$\text{(5.6)a} \qquad K_{\omega\mu\lambda} = 2\nabla_{\lambda}\, k_{\omega\mu} = 4a_{\lambda} a_{[\omega} u'_{\mu]},$$

$$\text{(5.6)b} \qquad k_{(\mu}{}^{\beta} K_{\lambda)\nu\beta} = 2(a_{(\mu} u^{\beta} - u_{(\mu} a^{\beta})\,(a_{\lambda)} u'_{\nu} - u'_{\lambda)} a_{\nu}) a_{\beta} = 0.$$

Therefore, we have according to Corollary II (7.1) (or by virtue of II (18.3)

$$\text{(5.7)} \qquad U^{\nu}{}_{\lambda\mu} = 0.$$ [1]

On the other hand, the equation I (3.9)b reduces by virtue of (5.4) to

$$*k^{\lambda\nu} = k^{\lambda\nu}$$

so that, by virtue of (5.4) and Theorems II (6.3) and I (14.1), or according to (5.7), II (7.23) and (5.5)b we have

$$S_{\lambda} = 0.$$

Hence, (2.8)a is satisfied, and the equations (3.13) reduce according to (5.7) and (2.8)a to

$$\text{(5.8)a} \qquad \partial_{[\omega} X_{\mu]} = \tfrac{1}{2}\nabla_{\alpha} K_{\omega\mu}{}^{\alpha}$$

$$\text{(5.8)b} \qquad 0 = H_{\mu\lambda} - \tfrac{1}{4} K_{\alpha\mu}{}^{\beta} K_{\beta\lambda}{}^{\alpha}.$$

Remembering (5.2)c, (5.6)a and (5.1) we obtain

$$\nabla_{\alpha}\, K_{\omega\mu}{}^{\alpha} = 4\partial_{\alpha} a^{\alpha} a_{[\omega} u'_{\mu]} = 4a^{\alpha} a_{[\omega} u''_{\mu]} a_{\alpha} = 0$$
$$K_{\alpha\mu}{}^{\beta} K_{\beta\lambda}{}^{\alpha} = 16 a^{\beta} a^{\alpha} a_{[\alpha} u'_{\mu]} a_{[\beta} u'_{\lambda]} = 0.$$

Therefore, the field equations are satisfied by $X_{\lambda} = \partial_{\lambda} X$ and $g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$, where $h_{\lambda\mu}$ and $k_{\lambda\mu}$ are defined by (5.1) and (5.3). The solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ of (2.7) is given by II (3.1), and this equation reduces in our case to

$$\text{(5.9)a} \qquad \Gamma_{\lambda}{}^{\nu}{}_{\mu} = S_{\lambda\mu}{}^{\nu}$$

where $S_{\lambda\mu}{}^{\nu}$ is given by II (17.3). Even this latter equation reduces considerably in our case because of

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[1] Another solution with (5.7) is in Ingraham, R., (1). The author derives it assuming the existence of vectors which are covariant constant with respect to $\{\lambda{}^{\nu}{}_{\mu}\}$. The existence of a covariant constant vector with respect to a non-symmetric connection is required in Stephenson, G., (1).

$$K_{\omega\alpha\beta}k_{\nu}{}^{\beta} = 4a_{\beta}a_{[\omega}u'{}_{\alpha]}(a_{\nu}u^{\beta} - u_{\nu}a^{\beta}) = 0.$$

Hence, we have by virtue of (5.9)a and II (17.3)

$$(5.9)\text{b} \qquad \Gamma_{\lambda}{}^{\nu}{}_{\mu} = \tfrac{1}{2}K_{\lambda\mu}{}^{\nu} = 2a^{\nu}a_{[\lambda}u'{}_{\mu]}.$$

NOTE. It will be shown later that this particular case of $g_{\lambda\mu}$ describes the plane wave in the electromagnetic theory of light.

Another solution of (2.7) and (2.8) is exhibited by Hlavatý, V., (13) in terms of polar coordinates

$$r \overset{\text{def}}{=} x^{\mathrm{I}},\ \theta \overset{\text{def}}{=} x^{\mathrm{II}},\ \varphi \overset{\text{def}}{=} x^{\mathrm{III}},\ t \overset{\text{def}}{=} x^{\mathrm{IV}},$$

$$(0 \leq \theta \leq \pi).$$

It consists of the vector $X_{\lambda} = \partial_{\lambda}X$ and of the tensor

$$(5.10)\text{a} \quad ((g_{\lambda\mu})) = \left(\begin{pmatrix} -1 & 0 & 0 & k_{\mathrm{I\,IV}} \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ k_{\mathrm{IV\,I}} & 0 & 0 & h_{\mathrm{IV\,IV}} \end{pmatrix}\right)$$

where

$$(5.10)\text{b} \qquad k_{\mathrm{I\,IV}} = \varepsilon\sqrt{|BA|}\,r^{-2}, \quad h_{\mathrm{IV\,IV}} = B(1 - Ar^{-4})$$

$$(\varepsilon = \pm 1)$$

and B is a positive constant, while A is a negative constant. The solution of (2.7) is given by [1]

$$(5.11) \qquad \Gamma_{\mathrm{II}}{}^{\mathrm{I}}{}_{\mathrm{II}} = -r,\ \Gamma_{\mathrm{III}}{}^{\mathrm{I}}{}_{\mathrm{III}} = -r\sin^2\theta,$$

$$\Gamma_{\mathrm{IV}}{}^{\mathrm{I}}{}_{\mathrm{IV}} = h_{\mathrm{IV\,IV}}(\ln g\sqrt{h_{\mathrm{IV\,IV}}})'$$

$$\Gamma_{\mathrm{III}}{}^{\mathrm{II}}{}_{\mathrm{III}} = -\sin\theta\cos\theta,$$

$$\Gamma_{\mathrm{I}}{}^{\mathrm{II}}{}_{\mathrm{II}} = \Gamma_{\mathrm{II}}{}^{\mathrm{II}}{}_{\mathrm{I}} = \Gamma_{\mathrm{I}}{}^{\mathrm{III}}{}_{\mathrm{III}} = \Gamma_{\mathrm{III}}{}^{\mathrm{III}}{}_{\mathrm{I}} = r^{-1}$$

$$\Gamma_{\mathrm{II}}{}^{\mathrm{III}}{}_{\mathrm{III}} = \Gamma_{\mathrm{III}}{}^{\mathrm{III}}{}_{\mathrm{II}} = \operatorname{cotg}\theta$$

$$\Gamma_{\mathrm{I}}{}^{\mathrm{I}}{}_{\mathrm{IV}} = -\Gamma_{\mathrm{IV}}{}^{\mathrm{I}}{}_{\mathrm{I}} = -2\Gamma_{\mathrm{II}}{}^{\mathrm{II}}{}_{\mathrm{IV}} = 2\Gamma_{\mathrm{IV}}{}^{\mathrm{II}}{}_{\mathrm{II}} = -2\Gamma_{\mathrm{III}}{}^{\mathrm{III}}{}_{\mathrm{IV}} =$$

$$= 2\Gamma_{\mathrm{IV}}{}^{\mathrm{III}}{}_{\mathrm{III}} = 2r^{-1}k_{\mathrm{I\,IV}}$$

$$\text{the remaining } \Gamma_{\lambda}{}^{\nu}{}_{\mu} = 0.$$

In subsequent considerations we shall use these particular solutions of (2.7) and (2.8) for constructing a general solution of this system.

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[1] Primes indicate derivatives with respect to r.

6. General solution of (2.8). In this section we start with a particular solution ${}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ of the system (2.8) in order to get its general solution. We could use either one of the two previous particular solutions. However, we shall generalize the problem slightly, assuming that ${}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is a solution of (2.8)b,c not requiring, in particular, that ${}^0\Gamma_{[\alpha}{}^{\alpha}{}_{\mu]} = 0$. All objects defined by means of ${}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ will be labeled by ^{0}. Thus, in particular, we define

$$(6.1) \qquad {}^0T_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=} {}^0S_{\lambda\mu}{}^{\nu} + {}^0U^{\nu}{}_{\lambda\mu}.$$

THEOREM (6.1). *Let* ${}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ *be a particular solution of* (2.8)b,c. *A necessary and sufficient condition that*

$$(6.2) \qquad \Gamma_{\lambda}{}^{\nu}{}_{\mu} = {}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu} + Q_{\lambda\mu}{}^{\nu}$$

be a solution of (2.8)a,b,c *is: The tensor* $Q_{\lambda\mu}{}^{\nu}$ *is a solution of*

$$(6.3) \qquad {}^0S_{\lambda} + Q_{[\lambda\alpha]}{}^{\alpha} = 0,$$

$$(6.4) \qquad \begin{aligned} \partial_{[\mu} Z_{\lambda]} = {}&{}^0D_{\mu} Q_{\lambda} + 2\,{}^0S_{\beta\mu}{}^{\alpha} Q_{\lambda\alpha}{}^{\beta} \\ &+ Q_{\alpha\mu}{}^{\beta} Q_{\lambda\beta}{}^{\alpha} \\ &- ({}^0D_{\alpha} Q_{\lambda\mu}{}^{\alpha} + Q_{\alpha} Q_{\lambda\mu}{}^{\alpha}). \end{aligned}$$

Here Z_{λ} *is a suitably chosen vector, and*

$$(6.5) \qquad Q_{\lambda} \overset{\text{def}}{=} Q_{\lambda\alpha}{}^{\alpha}.$$

PROOF. Using (6.2) and (1.2) we obtain

$$(6.6) \qquad \begin{aligned} R_{\omega\mu\lambda}{}^{\nu} = {}&{}^0R_{\omega\mu\lambda}{}^{\nu} + \\ &+ 2[{}^0D_{[\mu} Q_{|\lambda|\omega]}{}^{\nu} + {}^0S_{\omega\mu}{}^{\alpha} Q_{\lambda\alpha}{}^{\nu} + Q_{\alpha[\mu}{}^{\nu} Q_{|\lambda|\omega]}{}^{\alpha}], \end{aligned}$$

so that

$$(6.7) \qquad \begin{aligned} R_{\mu\lambda} = {}&{}^0R_{\mu\lambda} + {}^0D_{\mu} Q_{\lambda} + 2\,{}^0S_{\beta\mu}{}^{\alpha} Q_{\lambda\alpha}{}^{\beta} + Q_{\alpha\mu}{}^{\beta} Q_{\lambda\beta}{}^{\alpha} \\ &- ({}^0D_{\alpha} Q_{\lambda\mu}{}^{\alpha} + Q_{\alpha} Q_{\lambda\mu}{}^{\alpha}). \end{aligned}$$

The condition (6.4) follows from (2.8)b,c and (6.7). The condition (6.3) follows from (6.2) and (2.8)a.

NOTE. This Theorem is very handy for finding new solutions by means of ${}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. An application of this kind is exhibited by Hlavatý, V., (13): One starts with the connection ${}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ given by (5.11) and finds a connection

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = {}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu} + k_{\lambda\mu} p^{\nu}$$

where the conditions (6.3) and (6.4) define the vector p^ν. However, the actual computation is uninteresting. The final result for p^ν is

$$p^\nu\left(0,\ \frac{c}{\sin\theta},\ p^{\rm III}(r,t),\ 0\right)$$

$$(c = \text{const}).$$

THEOREM (6.2). *Let* ${}^0\Gamma_\lambda{}^\nu{}_\mu$ *be a solution of* (2.8). *Let* $\Gamma_\lambda{}^\nu{}_\mu$ *be another solution of* (2.8) *with the following properties:*

a) *it has the same skew symmetric part as* ${}^0\Gamma_\lambda{}^\nu{}_\mu$,

b) *it has the same autoparallel lines as* ${}^0\Gamma_\lambda{}^\nu{}_\mu$.

Then

$$\Gamma_\lambda{}^\nu{}_\mu = {}^0\Gamma_\lambda{}^\nu{}_\mu + \delta^\nu_\lambda p_\mu + \delta^\nu_\mu p_\lambda \tag{6.8}$$

where the vector p_λ *has to satisfy the conditions*

$$ {}^0D_{(\mu}p_{\lambda)} = p_\mu p_\lambda \tag{6.9a}$$

$$ \partial_{[\omega}({}^0S_{\mu\lambda]}{}^\alpha p_\alpha) = 0. \tag{6.9b}$$

The set of all vectors p_λ *satisfying these conditions is not an empty set.*

PROOF. It is very well known that the conditions a) and b) are satisfied by the connection (6.2) if and only if

$$Q_{\lambda\mu}{}^\nu = Q_{(\lambda\mu)}{}^\nu = \delta^\nu_\lambda p_\mu + \delta^\nu_\mu p_\lambda, \tag{6.10}$$

where p_λ is an arbitrary vector. The equations (6.3) are satisfied because of ${}^0S_\lambda = 0$ and by virtue of (6.10). The equations (6.4) reduce to

$$4\,{}^0D_\mu p_\lambda - {}^0D_\lambda p_\mu - 3p_\mu p_\lambda + 2\,{}^0S_{\lambda\mu}{}^\alpha p_\alpha = \partial_{[\mu}Z_{\lambda]}$$

and may be split in (6.9)a and

$$5\,{}^0D_{[\mu}p_{\lambda]} + 2\,{}^0S_{\lambda\mu}{}^\alpha p_\alpha = \partial_{[\mu}Z_{\lambda]}.$$

The last equation is equivalent to

$$5\partial_{[\mu}p_{\lambda]} - 3\,{}^0S_{\lambda\mu}{}^\alpha p_\alpha = \partial_{[\mu}Z_{\lambda]},$$

and (6.9)b are its integrability conditions. In order to prove the last statement we start again with the connection ${}^0\Gamma_\lambda{}^\nu{}_\mu$ given by (5.11). Then a simple inspection shows that the vector

$$p_\lambda\left(0,\ 0,\ 0,\ \frac{1}{c-t}\right),\quad (c = \text{const})$$

satisfies the conditions (6.9). Hence, the set of all vectors satisfying these conditions is not an empty set.

THEOREM (6.3). *Let* ${}^0\Gamma_\lambda{}^\nu{}_\mu$ *be a solution of* (2.8)b,c *which does not satisfy* (2.8)a. *Then*

$$\Gamma_\lambda{}^\nu{}_\mu = {}^0\Gamma_\lambda{}^\nu{}_\mu + \tfrac{2}{3}\,{}^0S_\mu \delta_\lambda^\nu \tag{6.11}$$ [1]

is a solution of (2.8)a,b,c.

PROOF. Put

$$Q_{\lambda\mu}{}^\nu = \delta_\lambda^\nu q_\mu, \quad Q_\lambda = q_\lambda.$$

Then (6.3) reduces to

$$ {}^0S_\lambda - \tfrac{3}{2}q_\lambda = 0. \tag{6.12}$$

The equations (6.4) reduce to

$$\partial_{[\mu} Z_{\lambda]} = 2({}^0D_{[\mu} q_{\lambda]} + {}^0S_{\lambda\mu}{}^\alpha q_\alpha) = 2\partial_{[\mu} q_{\lambda]}. \tag{6.13}$$

Hence,

$$q_\lambda = \tfrac{2}{3}{}^0S_\lambda = \tfrac{1}{2}Z_\lambda$$

solves our problem. [2]

7. Algebraic solution $\Gamma_\lambda{}^\nu{}_\mu$ of (2.8).

In this section we shall exhibit a device for solving the field equations (2.8) by a purely algebraic method. To this effect we shall first prove the following

THEOREM (7.1). *Let* $Z_{\omega\mu\lambda}{}^\nu$ *be an arbitrary tensor. A sufficient condition that the curvature tensor* $R_{\omega\mu\lambda}{}^\nu$ *of the connection* (6.2) *satisfy the equation*

$$R_{\omega\mu\lambda}{}^\nu = {}^0R_{\omega\mu\lambda}{}^\nu + 2Z_{[\omega\mu]\lambda}{}^\nu \tag{7.1}$$

is: The tensor $Q_{\lambda\mu}{}^\nu$ *is a solution of*

$${}^0D_\mu Q_{\lambda\omega}{}^\nu + Q_{\alpha\mu}{}^\nu Q_{\lambda\omega}{}^\alpha + {}^0T_{\omega\mu}{}^\alpha Q_{\lambda\alpha}{}^\nu = Z_{\omega\mu\lambda}{}^\nu. \tag{7.2}$$

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[1] A similar equation is also considered in Einstein, A. — Kaufman, B., (1); Horvath, J., (1). *Cf.* also Winogradski, J., (2); Husain, S. G., and Mishra, R. S., (1). The connection $\Gamma_\lambda{}^\nu{}_\mu$ defined by (6.11) is invariant with respect to the transformation

$${}^0\tilde{\Gamma}_\lambda{}^\nu{}_\mu = {}^0\Gamma_\lambda{}^\nu{}_\mu - 2\delta_\lambda^\nu p_\mu.$$

[2] For the existence of a general solution of (2.8) see Lichnerowicz, A., (1), (2), (3), (4). For particular solutions see footnotes on pp. 50 and 71, and Freistadt, H., (1); Narlikar, V. — Vaidya, P., (1); Papapetrou, A. — Schrödinger, E., (1); Rickayzen, G. — Kursunoğlu, P., (1); Wyman, M., (1); Udeschini, P., (8).

PROOF. Substituting from (6.2) into (1.2) one obtains

$$R_{\omega\mu\lambda}{}^{\nu} = {}^{0}R_{\omega\mu\lambda}{}^{\nu} \tag{7.3}$$
$$+ 2[{}^{0}D_{[\mu}Q_{|\lambda|\omega]}{}^{\nu} + {}^{0}S_{\omega\mu}{}^{\alpha}Q_{\lambda\alpha}{}^{\nu} + Q_{\alpha[\mu}{}^{\nu}Q_{|\lambda|\omega]}{}^{\alpha}].$$

On the other hand, (7.2) yields

$$ {}^{0}D_{[\mu}Q_{|\lambda|\omega]}{}^{\nu} + Q_{\alpha[\mu}{}^{\nu}Q_{|\lambda|\omega]}{}^{\alpha} + {}^{0}S_{\omega\mu}{}^{\alpha}Q_{\lambda\alpha}{}^{\nu} = Z_{[\omega\mu]\lambda}{}^{\nu}. \tag{7.4}$$

Substituting from (7.4) into (7.3) we obtain (7.1).

In the next Theorem we shall use a set of algebraic equations in $Q_{\lambda\mu}{}^{\nu}$

$$E_0,\ E_1,\ E_2,\ \ldots \tag{7.5}$$

constructed in the following way:

a) We denote by E_0 the system (6.3).

b) Differentiating E_0 and eliminating from the result ${}^{0}D_{\omega}Q_{\lambda\mu}{}^{\nu}$ by means of (7.2), we obtain

$$ {}^{0}D_{\mu}{}^{0}S_{\lambda} - Q_{\alpha\mu}{}^{\beta}Q_{[\lambda\beta]}{}^{\alpha} - {}^{0}T_{[\beta|\mu|}{}^{\alpha}Q_{\lambda]\alpha}{}^{\beta} = - Z_{[\alpha|\mu|\lambda]}{}^{\alpha}. \tag{7.6a}$$

Differentiating (7.2) and alternating indices, we obtain a set of equations for ${}^{0}D_{[\xi}{}^{0}D_{\mu]}Q_{\lambda\omega}{}^{\nu}$. Using (1.2) and (7.2) we eliminate ${}^{0}D_{\omega}Q_{\lambda\mu}{}^{\nu}$ from these equations. Let the resulting set be denoted by

$$E_{\xi\omega\mu\lambda}{}^{\nu} = 0. \tag{7.6b}$$

We denote by E_1 the set consisting of (7.6)a,b.

c) Differentiating E_1 and eliminating from the result ${}^{0}D_{\omega}Q_{\lambda\mu}{}^{\nu}$ by means of (7.2), we obtain E_2.

d) Continuing this procedure we finally get (7.5).

THEOREM (7.2). *In order that the system of equations* (6.3) *and* (7.2) *admit a solution* $Q_{\lambda\mu}{}^{\nu}$, *it is necessary and sufficient that there exist a positive integer* $N \leq n^3$ [1] *such that the equations of the set*

$$E_0,\ E_1,\ E_2,\ \ldots,\ E_N \tag{7.7}$$

are compatible for all values of the x's in some domain, and that the equations of the set E_{N+1} *be satisfied because of* (7.7). *If* p *is the number of independent equations in the set* (7.7), *the solution involves* $n^3 - p$ *arbitrary constants.*

PROOF. This is almost an exact wording of a more general Theorem in Eisenhart, L., (2) where its proof may be found.

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[1] n^3 is the number of components of $Q_{\lambda\mu}{}^{\nu}$ in an n dimensional space.

NOTE. The importance of this Theorem lies in the fact that *it yields the solution by an elementary algebraic method.* [1] Whenever this solution exists, it is a solution of the algebraic system (7.7).

THEOREM (7.3). *Let* $'Z_{\omega\mu\lambda}{}^{\nu}$ *be an arbitrary tensor satisfying the condition*

$$'Z_{[\alpha\mu]\lambda}{}^{\alpha} = 0 \tag{7.8}$$

and put

$$Z_{\omega\mu\lambda}{}^{\nu} \overset{\text{def}}{=} \frac{1}{n-1}\,\delta^{\nu}_{\omega}\,\partial_{[\mu}Z_{\lambda]} + {}'Z_{\omega\mu\lambda}{}^{\nu}. \tag{7.9}$$

If the conditions of the previous Theorem are satisfied for the special form (7.9) *of* $Z_{\omega\mu\lambda}{}^{\nu}$, *then the solution* $Q_{\lambda\mu}{}^{\nu}$ *of* (6.3) *and* (7.2), *obtained by purely algebraic methods, is also a solution of* (6.3) *and* (6.4).

PROOF. Equations (7.4) follow from (7.2). If we put $\omega = \nu$ in (7.4) and sum, we obtain

$$\begin{aligned} {}^{0}D_{\mu}Q_{\lambda} - {}^{0}D_{\alpha}Q_{\lambda\mu}{}^{\alpha} &+ Q_{\alpha\mu}{}^{\beta}Q_{\lambda\beta}{}^{\alpha} \\ - Q_{\alpha}Q_{\lambda\mu}{}^{\alpha} &+ 2\,{}^{0}S_{\beta\mu}{}^{\alpha}Q_{\lambda\alpha}{}^{\beta} = 2Z_{[\alpha\mu]\lambda}{}^{\alpha}. \end{aligned} \tag{7.10)a$$

On the other hand, we have according to (7.8) and (7.9)

$$2Z_{[\alpha\mu]\lambda}{}^{\alpha} = \partial_{[\mu}Z_{\lambda]} \tag{7.10)b$$

so that (7.10)a reduces to (6.4).

NOTE I. Having obtained the solution $Q_{\lambda\mu}{}^{\nu}$ of (6.3) and (6.4) in a purely algebraic way, we substitute it into (6.2) and obtain in this way the *solution* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ *of* (2.8) *by a purely algebraic method.*

NOTE II. We replaced the underdetermined system (6.4) of n^2 equations by an overdetermined system (7.2) of n^4 equations. However, this overdeterminacy is offset by the arbitrary tensor $Z_{\omega\mu\lambda}{}^{\nu}$ involved in (7.2). According to (7.8) $'Z_{\omega\mu\lambda}{}^{\nu}$ represents $n^4 - n^2$, arbitrarily chosen functions involved in the set (7.2) of n^4 equations. Therefore, this set could be looked upon as representing only $n^4 - (n^4 - n^2) = n^2$ conditions for $Q_{\lambda\mu}{}^{\nu}$ as does the set (6.4) with which we started.

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[1] The method which yields a solution of a completely integrable system by means of power series is considered here (and in subsequent results) as an elementary algebraic method.

8. Algebraic solution $\Gamma_\lambda{}^\nu{}_\mu$, $g_{\lambda\mu}$ of (2.7) and (2.8). Upon substituting the solution $\Gamma_\lambda{}^\nu{}_\mu$ of (2.8) into (2.7), we obtain an overdetermined set of n^3 differential equations of the first order for n^2 unknowns $g_{\lambda\mu}$. It is not difficult to derive from (1.6)a a set of algebraic homogeneous equations of the first degree for $g_{\alpha\beta}$ which yields the solution $g_{\lambda\mu}$ of (2.7). However, we shall use a slightly different approach which seems to us more advantageous.

Assume that we have a particular solution ${}^0g_{\lambda\mu}$, ${}^0\Gamma_\lambda{}^\nu{}_\mu$ of (2.7) and (2.8) (see, for instance, the solutions exhibited in section five). The question arises whether we could express $g_{\lambda\mu}$ by means of ${}^0g_{\lambda\mu}$ in a way similar to that by which we expressed $\Gamma_\lambda{}^\nu{}_\mu$ by means of ${}^0\Gamma_\lambda{}^\nu{}_\mu$. In order to answer this question we put $g_{\lambda\mu}$ in the form

$$(8.1) \qquad g_{\lambda\mu} = {}^0g_{\lambda\mu} + X_{\lambda\mu}$$

and find $X_{\lambda\mu}$.

THEOREM (8.1). *Denote by* ${}^0g_{\lambda\mu}$, ${}^0\Gamma_\lambda{}^\nu{}_\mu$ *a particular solution of* (2.7) *and* (2.8). *In order that there exist a tensor* (8.1) *which together with the* $\Gamma_\lambda{}^\nu{}_\mu$ *is a solution of* (2.7) *and* (2.8), *it is necessary and sufficient that the system*

$$(8.2) \qquad {}^0D_\omega X_{\lambda\mu} = X_{\alpha\beta} Q^{\alpha\beta}_{\omega\lambda\mu} + G_{\omega\lambda\mu}$$

admit a solution. Here

$$(8.3)\text{a} \qquad Q^{\alpha\beta}_{\omega\lambda\mu} \stackrel{\text{def}}{=} Q_{\lambda\omega}{}^\alpha \delta^\beta_\mu + \delta^\alpha_\lambda(Q_{\omega\mu}{}^\beta + 2\,{}^0S_{\omega\mu}{}^\beta)$$

$$(8.3)\text{b} \qquad G_{\omega\lambda\mu} \stackrel{\text{def}}{=} {}^0g_{\alpha\beta}(Q^{\alpha\beta}_{\omega\lambda\mu} - 2\,{}^0S_{\omega\mu}{}^\beta \delta^\alpha_\lambda).$$

PROOF. We have according to (6.2) and (8.1)

$$D_\omega g_{\lambda\mu} = {}^0D_\omega({}^0g_{\lambda\mu} + X_{\lambda\mu}) - Q_{\lambda\omega}{}^\alpha({}^0g_{\alpha\mu} + X_{\alpha\mu}) - Q_{\mu\omega}{}^\alpha({}^0g_{\lambda\alpha} + X_{\lambda\alpha})$$

$$2S_{\omega\mu}{}^\alpha g_{\lambda\alpha} = 2\,{}^0S_{\omega\mu}{}^\alpha({}^0g_{\lambda\alpha} + X_{\lambda\alpha}) + 2Q_{[\omega\mu]}{}^\alpha({}^0g_{\lambda\alpha} + X_{\lambda\alpha})$$

so that

$$(8.4) \qquad D_\omega g_{\lambda\mu} - 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha} = {}^0D_\omega {}^0g_{\lambda\mu} - 2\,{}^0S_{\omega\mu}{}^\alpha\, {}^0g_{\lambda\alpha} + {}^0D_\omega X_{\lambda\mu} - 2\,{}^0S_{\omega\mu}{}^\alpha X_{\lambda\alpha} - ({}^0g_{\alpha\beta} + X_{\alpha\beta})\,(\delta^\beta_\mu Q_{\lambda\omega}{}^\alpha + \delta^\alpha_\lambda Q_{\omega\mu}{}^\beta).$$

According to our assumption, ${}^0g_{\lambda\mu}$ is a solution of

$${}^0D_\omega {}^0g_{\lambda\mu} - 2\,{}^0S_{\omega\mu}{}^\alpha\, {}^0g_{\lambda\alpha} = 0.$$

Hence, a necessary and sufficient condition for $g_{\lambda\mu}$ to satisfy (2.7) is that (8.4) reduce to (8.2).

In the next Theorem we shall use the following abbreviations:

$$\text{(8.5)a} \qquad \mathrm{A}^{\alpha\beta}_{\xi\omega\lambda\mu} \overset{\text{def}}{=} \tfrac{1}{2}{}^0R_{\xi\omega\lambda}{}^{\alpha}\delta^{\beta}_{\mu} + \tfrac{1}{2}{}^0R_{\xi\omega\mu}{}^{\beta}\delta^{\alpha}_{\lambda}$$
$$- {}^0S_{\omega\xi}{}^{\gamma}Q^{\alpha\beta}_{\gamma\lambda\mu} - \delta^{\nu}_{[\xi}Q^{\gamma\delta}_{\omega]\lambda\mu}Q^{\alpha\beta}_{\nu\gamma\delta} - {}^0D_{[\xi}Q^{\alpha\beta}_{\omega]\lambda\mu},$$

$$\text{(8.5)b} \qquad A^{\alpha\beta}_{\xi_{p+1}\cdots\xi_1\omega\lambda\mu} \overset{\text{def}}{=} {}^0D_{\xi_{p+1}}A^{\alpha\beta}_{\xi_p\cdots\xi_1\omega\lambda\mu} + Q^{\alpha\beta}_{\xi_{p+1}\gamma\delta}A^{\gamma\delta}_{\xi_p\cdots\xi_1\omega\lambda\mu}$$

$$\text{(8.5)c} \qquad B_{\xi\omega\lambda\mu} \overset{\text{def}}{=} G_{\alpha\gamma\delta}({}^0S_{\omega\xi}{}^{\alpha}\delta^{\gamma\delta}_{\lambda\mu} + \delta^{\alpha}_{[\xi}Q^{\gamma\delta}_{\omega]\lambda\mu}) + {}^0D_{[\xi}G_{\omega]\lambda\mu}$$

$$\text{(8.5)d} \qquad B_{\xi_{p+1}\cdots\xi_1\omega\lambda\mu} \overset{\text{def}}{=} {}^0D_{\xi_{p+1}}B_{\xi_p\cdots\xi_1\omega\lambda\mu} - G_{\xi_{p+1}\gamma\delta}A^{\gamma\delta}_{\xi_p\cdots\xi_1\omega\lambda\mu}$$
$$(p = 1, 2, \ldots).$$

Moreover, we denote by $((AB))$ a rectangular matrix of $n^2 + 1$ columns. The elements of its first n^2 columns are $A^{\alpha\beta}_{\xi_{p+1}\cdots\xi_1\omega\lambda\mu}$; the elements of its last column are $B_{\xi_{p+1}\cdots\xi_1\omega\lambda\mu}$. The first n^2 columns are labeled by the superscripts $\alpha\beta$ of the A's. The rows of $((AB))$ are labeled by the subscripts $\xi_{p+1}\ldots\xi_1\omega\lambda\mu$ of the A's and B's, $p = 1, 2, \ldots$. The matrix which one obtains by leaving out the last column of $((AB))$ will be denoted by $((A))$.

THEOREM (8.2). *In order that the system* (8.2) *shall admit a solution, it is necessary and sufficient that the matrices* $((A))$ *and* $((AB))$ *of the nonhomogeneous linear system in* $X_{\alpha\beta}$

$$\text{(8.6)} \qquad X_{\alpha\beta}A^{\alpha\beta}_{\xi_p\cdots\xi_1\omega\lambda\mu} = B_{\xi_p\cdots\xi_1\omega\lambda\mu}$$
$$(p = 1, 2, \ldots)$$

be of the same rank $n^2 - q$ $(0 \leq q < n^2)$ *in a certain domain of the coordinates* x^{λ}. *If this condition is satisfied, then the solution* $X_{\lambda\mu}$ *involves* q *arbitrary constants.*

PROOF. Using (8.2) one obtains

$$\text{(8.7)a} \qquad {}^0D_{[\xi}{}^0D_{\omega]}X_{\lambda\mu} = {}^0D_{[\xi}(Q^{\alpha\beta}_{\omega]\lambda\mu}X_{\alpha\beta} + G_{\omega]\lambda\mu}).$$

The left-hand term may be transformed by means of (1.3)

$$\text{(8.7)b} \qquad {}^0D_{[\xi}{}^0D_{\omega]}X_{\lambda\mu} =$$
$$= \tfrac{1}{2}{}^0R_{\xi\omega\lambda}{}^{\alpha}X_{\alpha\mu} + \tfrac{1}{2}{}^0R_{\xi\omega\mu}{}^{\alpha}X_{\lambda\alpha} + {}^0S_{\xi\omega}{}^{\alpha}\,{}^0D_{\alpha}X_{\lambda\mu}.$$

Eliminating ${}^0D_{\alpha}X_{\lambda\mu}$ from (8.7)a,b and (8.2) we obtain (8.6) for $p = 1$. Differentiating this equation and eliminating ${}^0D_{\omega}X_{\lambda\mu}$ by means of (8.2), we get (8.6) for $p = 2$. Continuing in the same way we obtain (8.6) for $p = 1, 2, 3, \ldots$. Now, our Theorem is only an application of Eisenhart's Theorem which we mentioned earlier.

NOTE I. The importance of this Theorem lies in the fact that, if $X_{\lambda\mu}$ defined by (8.2) exists, it may be obtained by *an elementary algebraic method*. Recapitulating our results we see that *the knowledge of a particular solution* $^0g_{\lambda\mu}$, $^0\Gamma_\lambda{}^\nu{}_\mu$ *of* (2.7) *and* (2.8) *enables us to find other solutions* $g_{\lambda\mu}$, $\Gamma_\lambda{}^\nu{}_\mu$ *of these equations by elementary algebraic methods.*

NOTE II. For a solution of (2.8) by means of Cauchy's problem, see Lichnerowicz, A., (2), (3), (4).

9. Conformal change. In the concluding sections, nine through eleven, of this chapter we shall investigate the conformal change of the tensor $g_{\lambda\mu}$ in relation to the connection $\Gamma_\lambda{}^\nu{}_\mu$ given by (2.7) and with the vector S_λ involved in (2.8)a.

Let $\Omega = \Omega(x)$ be an arbitrary function of position with at least two derivatives. Put

$$'g_{\lambda\mu} = e^{\Omega} g_{\lambda\mu}. \tag{9.1}$$

This conformal transformation enforces a change of the connection. This change can always be expressed as follows:

$$'\Gamma_\lambda{}^\nu{}_\mu = \Gamma_\lambda{}^\nu{}_\mu + q_{\lambda\mu}{}^\nu \tag{9.2a}$$

where the tensor $q_{\lambda\mu}{}^\nu$ has to be found. Here, of course, $'\Gamma_\lambda{}^\nu{}_\mu$ is the same function of $'g_{\lambda\mu}$ as $\Gamma_\lambda{}^\nu{}_\mu$ is of $g_{\lambda\mu}$. In subsequent considerations we use the tensors $s_{\lambda\mu}{}^\nu$ and $u^\nu{}_{\lambda\mu}$ defined by

$$s_{\lambda\mu}{}^\nu \overset{\text{def}}{=} q_{[\lambda\mu]}{}^\nu, \quad u^\nu{}_{\lambda\mu} \overset{\text{def}}{=} q_{(\lambda\mu)}{}^\nu \tag{9.3}$$

and the tensors $a_{\omega\mu\lambda}$, $b_{\omega\mu\lambda}$ defined by

$$a_{\omega\mu\lambda} \overset{\text{def}}{=} w_\omega h_{\lambda\mu} + w_\mu h_{\omega\lambda} - w_\lambda h_{\mu\omega} \tag{9.4a}$$

$$b_{\omega\mu\lambda} \overset{\text{def}}{=} w_\omega k_{\lambda\mu} + w_\mu k_{\omega\lambda} - w_\lambda k_{\mu\omega}, \tag{9.4b}$$

where

$$w_\alpha \overset{\text{def}}{=} \partial_\alpha \Omega. \tag{9.4c}$$

Using (9.3) we may write (9.2)a as

$$'\Gamma_\lambda{}^\nu{}_\mu = \Gamma_\lambda{}^\nu{}_\mu + s_{\lambda\mu}{}^\nu + u^\nu{}_{\lambda\mu}. \tag{9.2b}$$

We have now, besides $h_{\lambda\mu}$ and $h^{\lambda\nu}$, also the tensors

$$'h_{\lambda\mu} = e^{\Omega} h_{\lambda\mu}, \quad 'h^{\lambda\nu} = e^{-\Omega} h^{\lambda\nu}.$$

Throughout sections nine, ten and eleven we shall hold to the following agreement: If T is a function of $g_{\lambda\mu}$, then we denote by $'T$ the same function of $'g_{\lambda\mu}$. If, in particular, T is a tensor, so is $'T$, and our rule implies that the indices of T (of $'T$) be raised and/or lowered by means of $h^{\lambda\nu}$ and/or $h_{\lambda\mu}$ (by means of $'h^{\lambda\nu}$ and/or $'h_{\lambda\mu}$).

THEOREM (9.1). *The tensors* $s_{\lambda\mu}{}^{\nu}$ *and* $u^{\nu}{}_{\lambda\mu}$ *involved in* (9.2)b *satisfy the following relations:*

$$a_{\omega\mu\lambda} = 2u_{\lambda\omega\mu} - 4s_{\lambda(\omega}{}^{\alpha}k_{\mu)\alpha} \tag*{(9.5)a}$$

$$b_{\omega\mu\lambda} = 2s_{\omega\mu\lambda} + 4u_{\alpha\lambda[\omega}k_{\mu]}{}^{\alpha}. \tag*{(9.5)b}$$

PROOF. A simple calculation based on (9.2)a and (9.1) yields

$$\begin{aligned} &e^{-\Omega}('D_{\omega}{}'g_{\lambda\mu} - 2'S_{\omega\mu}{}^{\alpha}{}'g_{\lambda\alpha}) \\ &= D_{\omega}g_{\lambda\mu} - 2S_{\omega\mu}{}^{\alpha}g_{\lambda\alpha} + w_{\omega}h_{\lambda\mu} - 2s_{\omega(\mu}{}^{\alpha}k_{\lambda)\alpha} - 2u_{(\lambda\mu)\omega} \\ &\qquad + w_{\omega}k_{\lambda\mu} - 2s_{\omega[\mu\lambda]} - 2u^{\alpha}{}_{\omega[\mu}k_{\lambda]\alpha}. \end{aligned} \tag*{(9.6)}$$

The tensor $g_{\lambda\mu}$ satisfies (2.7) and the tensor $'g_{\lambda\mu}$ satisfies the relation

$$'D_{\omega}{}'g_{\lambda\mu} = 2'S_{\omega\mu}{}^{\alpha}\ 'g_{\lambda\alpha}. \tag*{(9.7)}$$

Therefore, (9.6) reduces to

$$w_{\omega}h_{\lambda\mu} = 2(s_{\omega(\mu}{}^{\alpha}k_{\lambda)\alpha} + u_{(\lambda\mu)\omega}) \tag*{(9.8)a}$$

$$w_{\omega}k_{\lambda\mu} = 2(s_{\omega[\mu\lambda]} + u^{\alpha}{}_{\omega[\mu}k_{\lambda]\alpha}). \tag*{(9.8)b}$$

The equations (9.5) follow from (9.4) and (9.8).

THEOREM (9.2). *The equations* (9.5) *are equivalent to*

$$2s_{\alpha\beta\gamma}X^{\alpha\beta\gamma}_{\omega\mu\nu} = k_{\omega\mu\nu} \tag*{(9.9)a}$$

$$\begin{aligned} &u_{\alpha\beta\gamma}\{\delta^{\alpha\beta\gamma}_{\nu(\lambda\mu)} - 2k_{(\mu}{}^{\beta}k_{\lambda)}{}^{\alpha}\delta^{\gamma}_{\nu} + 2k_{\nu}{}^{\alpha}k_{(\mu}{}^{\beta}\delta^{\gamma}_{\lambda)}\} \\ &\qquad = \tfrac{1}{2}a_{\lambda\mu\nu} - k_{(\lambda}{}^{\alpha}b_{\mu)\nu\alpha}. \end{aligned} \tag*{(9.9)b}$$ [1]

Here

$$\begin{aligned} k_{\omega\mu\nu} &\overset{\text{def}}{=} b_{\omega\mu\nu} - 2k_{[\omega}{}^{\alpha}a_{\mu]\nu\alpha} \\ &= 2w_{\alpha}k_{[\omega}{}^{\alpha}h_{\mu]\nu} - w_{\nu}k_{\omega\mu}, \end{aligned} \tag*{(9.10)}$$

and $X^{\alpha\beta\gamma}_{\omega\mu\nu}$ *is defined by* II (3.11).

---

[1] The coefficient of $U_{\alpha\beta\gamma}$ in II (7.2) is the same as the coefficient of $u_{\alpha\beta\gamma}$ in (9.9)b.

PROOF. The last part of (9.10) follows from the middle part and from (9.4). The equation (9.9)a ((9.9)b) is the result of the elimination of $u_{\alpha\beta\gamma}$ (of $s_{\alpha\beta\gamma}$) from (9.5).

In the next Theorem we use the tensor $Z^{\alpha\beta\gamma}_{\omega\mu\nu}$ defined by II (10.3).

THEOREM (9.3). *Consider the first two classes, and put*

(9.11)a $$l_{\omega\mu\nu} \overset{\text{def}}{=} Z^{\alpha\beta\gamma}_{\omega\mu\nu} k_{\alpha\beta\gamma} = l_{[\omega\mu]\nu}.$$

Then

(9.12)a $$2s_{\omega\mu\nu} = l_{\omega\mu\nu} + 3l_{[\alpha\beta\nu]} k_{\omega}{}^{\alpha} k_{\mu}{}^{\beta}$$

(9.12)b $$2u_{\nu\lambda\mu} = a_{\lambda\mu\nu} - 2k_{(\lambda}{}^{\gamma}\ l_{\mu)\nu\gamma}$$

so that

(9.13) $$'\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{\lambda}{}^{\nu}{}_{\mu} + \tfrac{1}{2}[a_{\lambda\mu}{}^{\nu} + l_{\lambda\mu}{}^{\nu} + 3l_{[\alpha\beta\gamma]} k_{\lambda}{}^{\alpha} k_{\mu}{}^{\beta} h^{\gamma\nu} \\ - 2k_{(\lambda}{}^{\alpha} l_{\mu)\gamma\alpha} h^{\gamma\nu}].$$

PROOF. The structures of II (3.12) and (9.9)a are the same. Therefore, we may apply the method of section II (11) for solving (9.9)a. Remembering in particular II (11.1) and II (10.6) we obtain (9.12)a and (9.11)a, where, of course, the nonholonomic components l_{xyz} of $l_{\omega\mu\nu}$ are given by

(9.11)b $$l_{xyz} = \frac{1}{\underset{xyz}{\lambda}} k_{xyz}.$$

Substituting from (9.12)a for $s_{\omega\mu\nu}$ into (9.5)a one obtains (9.12)b. The equation (9.13) follows from (9.12) and (9.2)b.

In subsequent considerations we shall need two Lemmas. The first one is

LEMMA (9.1). *We have for all three classes*

(9.14)a $$2s_{[\omega\mu\nu]} = k_{[\omega\mu\nu]} = -\ w_{[\nu} k_{\omega\mu]}$$

(9.14)b $$2u_{(\nu\lambda\mu)} = a_{(\lambda\mu\nu)} = w_{(\lambda} h_{\mu\nu)}.$$

PROOF. Consider the first two classes. The equation (9.12)a may be rewritten in nonholonomic components by means of (9.11)b and I (9.1)a (for $p = 1$). Then

$$6\,\underset{xyz}{\lambda}\, s_{[xyz]} = 3k_{[xyz]} + 3\underset{x}{\lambda}\underset{y}{\lambda} k_{[xyz]} + \\ + 3\underset{y}{\lambda}\underset{z}{\lambda} k_{[yzx]} + 3\underset{z}{\lambda}\underset{x}{\lambda} k_{[zxy]} = 3\,\underset{xyz}{\lambda}\, k_{[xyz]},$$

and this equation is equivalent to the first equation (9.14)a. The second equation (9.14)a follows from the first one and from (9.10). The equations (9.14)b follow directly from (9.4)a and (9.12)b. The proof for the third class will be given later.

In the second Lemma we also use nonholonomic components with respect to the nonholonomic frame exhibited in sections I (12) and I (13).

LEMMA (9.2). *For the third class we have*

$$k_{\boldsymbol{i}[\mathbf{24}]} = 0 \tag{9.15a}$$

$$k_{\mathbf{434}} = 0 \tag{9.15b}$$

and

$$k_{\nu}{}^{\beta} k_{(\mu}{}^{\alpha} k_{\lambda)}{}^{\gamma} k_{\alpha\beta\gamma} = 0. \tag{9.16}$$

PROOF. The equations (9.15) follow directly from the nonholonomic form of (9.10) and from I (12.3) and I (13.1). Then the proof of (9.16) follows the pattern of the proof of Lemma II (18.1).

THEOREM (9.4). *We have for the third class*

$$2s_{\omega\mu\nu} = k_{\omega\mu\nu} - 2k_{[\mu}{}^{\alpha} k_{\omega]\alpha\beta} k_{\nu}{}^{\beta} \tag{9.17a}$$

$$\begin{aligned} 2u_{\nu\lambda\mu} = a_{\lambda\mu\nu} + 2h_{\gamma\alpha\beta}[\delta^{\gamma\alpha\beta}_{\nu(\lambda\mu)} + \\ + 2\delta^{\alpha}_{\nu} k_{(\lambda}{}^{\gamma} k_{\mu)}{}^{\beta} - 2\delta^{\beta}_{(\lambda} k_{\mu)}{}^{\alpha} k_{\nu}{}^{\gamma}]. \end{aligned} \tag{9.17b}$$

Here

$$h_{\gamma\alpha\beta} \overset{\text{def}}{=} k_{\gamma(\alpha}{}^{\delta} k_{\beta)\delta}. \tag{9.18}$$

Therefore, the connection $'\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ *is given by*

$$\begin{aligned} '\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{\lambda}{}^{\nu}{}_{\mu} + \tfrac{1}{2} a_{\lambda\mu}{}^{\nu} + \tfrac{1}{2} k_{\lambda\mu}{}^{\nu} - k_{[\mu}{}^{\alpha} k_{\lambda]\alpha\beta} k^{\nu\beta} \\ + h_{\gamma\alpha\beta} h^{\omega\nu}[\delta^{\gamma}_{\omega}{}^{\alpha\beta}_{(\lambda\mu)} + 2\delta^{\alpha}_{\omega} k_{(\lambda}{}^{\gamma} k_{\mu)}{}^{\beta} - 2\delta^{\beta}_{(\lambda} k_{\mu)}{}^{\alpha} k_{\omega}{}^{\gamma}]. \end{aligned} \tag{9.19}$$

PROOF. The structure of II (3.12) and II (15.3) being the same as the structure of (9.9)a and (9.15), we obtain (9.17)a by replacing the tensor $K_{\omega\mu\nu}$ with $k_{\omega\mu\nu}$ in the solution II (17.3). The equation (9.17)b follows from (9.17)a, (9.5)a and (9.16). (*Cf.* also the proof of Theorem II (18.1)). The equation (9.19) follows from (9.17) and (9.2)b.

NOTE. Using (9.17) we may easily prove Lemma (9.1) for the third class by means of I (13.1)a,b. According to these latter equations the only nonholonomic component of $k_{[\nu}{}^{\beta}k_{\mu}{}^{\alpha}k_{\omega]\alpha\beta}$ which does not identically vanish is $k_{1[24]}$. This component is equal to zero because of (9.15). Therefore, we obtain from (9.17)a

$$2s_{[\omega\mu\nu]} = k_{[\omega\mu\nu]} - 2k_{[\nu}{}^{\beta}k_{\mu}{}^{\alpha}k_{\omega]\alpha\beta} = k_{[\omega\mu\nu]}.$$

The equation (9.14)b follows in a similar way from (9.17)b.

In the next section we shall derive some consequences of Theorems (9.3) and (9.4).

10. The vector $'S_\lambda$. In the next Theorem we shall use the vectors

$$S_\lambda \overset{\text{def}}{=} S_{\lambda\alpha}{}^{\alpha}, \quad 'S_\lambda \overset{\text{def}}{=} {}'S_{\lambda\alpha}{}^{\alpha}.$$

THEOREM (10.1). *The vector S_λ is transformed by the conformal change* (9.1) *as follows:*

$$'S_\lambda = S_\lambda + w_\alpha k_\lambda{}^{\alpha}. \tag{10.1}$$

PROOF. We have from (9.2)b

$$'S_{\lambda\mu}{}^{\nu} = S_{\lambda\mu}{}^{\nu} + s_{\lambda\mu}{}^{\nu} \tag{10.2)a}$$

so that

$$'S_\lambda = S_\lambda + s_{\lambda\alpha}{}^{\alpha}. \tag{10.2)b}$$

Consider the first two classes. The nonholonomic components of

$$s_\lambda \overset{\text{def}}{=} s_{\lambda\alpha}{}^{\alpha}$$

are

$$s_x = s_{xx'x} + s_{xzz'} + s_{xz'z}.\,^{1} \tag{10.3)a}$$

Therefore, we have according to (9.11)b, (9.12)a and II (9.1), II (14.1),

$$2s_x = \frac{1}{\underset{xx'x}{\lambda}}\, k_{xx'x} + \frac{1}{\underset{xzz'}{\lambda}}\, [k_{xzz'} + k_{xz'z} + 6\underset{x}{\lambda}\underset{z}{\lambda} k_{[xzz']}].$$

This expression reduces by virtue of (9.10) to

$$s_x = \underset{x}{\lambda} w_x$$

equivalent to

$$s_\lambda = w_\alpha k_\lambda{}^{\alpha}. \tag{10.3)b}$$

---

[1] The primed indices are defined in the proof of Theorem III (2.3). If the range of x is 1, 2 (or 3, 4), then the range of z is 3, 4 (or 1, 2).

Substituting this value into (10.2)b we obtain (10.1).

Consider now the third class. Then we have from I (13.3)

$$ {}^{(2)}k_\lambda{}^\lambda = 0 \qquad {}^{(3)}k_\lambda{}^\nu = 0 $$

so that according to (9.17)a and (9.10)

$$ \begin{aligned} (10.3)\text{b} \qquad 2s_\omega &= k_{\omega\alpha}{}^\alpha + 2k_{[\omega}{}^\alpha k_{\mu]\alpha\beta} k_\nu{}^\beta h^{\mu\nu} \\ &= k_{\omega\alpha}{}^\alpha = 4w_\alpha k_\omega{}^\alpha - w_\alpha k_\omega{}^\alpha - w_\alpha k_\omega{}^\alpha \\ &= 2w_\alpha k_\omega{}^\alpha. \end{aligned} $$

Hence, even in this case we obtain (10.1).

According to Theorem I (14.1) the tensor ${}^*h^{\lambda\nu}$ is of rank 4. Therefore, there is only one inverse tensor ${}^*\overset{-1}{h}_{\lambda\mu} = {}^*\overset{-1}{h}_{\mu\lambda}$ satisfying the condition

$$ (10.4) \qquad {}^*\overset{-1}{h}_{\lambda\mu} {}^*h^{\nu\mu} = \delta_\lambda^\nu. $$

We shall use this tensor in the next theorem.

THEOREM (10.2). *In order that* $'S_\lambda$ *vanish*

$$ (10.5) \qquad 'S_\lambda = 0, $$

it is necessary and sufficient that there exist a scalar Ω *such that*

$$ (10.6) \qquad \frac{1}{\sqrt{|\mathfrak{g}|}} {}^*\overset{-1}{h}_{\mu\nu} \partial_\lambda \sqrt{|\mathfrak{g}|}\, {}^*k^{\lambda\nu} = k_\mu{}^\alpha\, \partial_\alpha \Omega. $$

PROOF. Multiplying II (6.14) by ${}^*\overset{-1}{h}_{\mu\nu}$ we obtain

$$ -\sqrt{|\mathfrak{g}|}\, S_\mu = {}^*\overset{-1}{h}_{\mu\nu} \partial_\lambda \sqrt{|\mathfrak{g}|}\, {}^*k^{\lambda\nu} $$

so that (10.2)b reduces by virtue of (10.3)b to

$$ 'S_\mu \sqrt{|\mathfrak{g}|} = w_\alpha k_\mu{}^\alpha \sqrt{|\mathfrak{g}|} - {}^*\overset{-1}{h}_{\mu\nu} \partial_\lambda \sqrt{|\mathfrak{g}|}\, {}^*k^{\lambda\nu}. $$

The proof follows easily from this equation and (9.4)c.

In the case of the first class we have $k \neq 0$, and therefore, there is only one tensor $\overset{-1}{k}_\lambda{}^\mu$ satisfying the condition

$$ (10.7) \qquad \overset{-1}{k}_\lambda{}^\mu k_\mu{}^\nu = \delta_\lambda^\nu = k_\lambda{}^\mu \overset{-1}{k}_\mu{}^\nu. $$

We shall use it in the next Theorem.

THEOREM (10.3). *In order that* (10.5) *hold for the first class it is necessary and sufficient that*

$$(10.8)\text{a} \qquad \partial_{[\mu}\left[\overset{-1}{k}_{\omega]}{}^{\alpha}\,{}^{*}\overset{-1}{h}_{\alpha\beta}(\partial_{\gamma}\sqrt{|\mathfrak{g}|}\,{}^{*}k^{\gamma\beta})\,\frac{1}{\sqrt{|\mathfrak{g}|}}\right] = 0.$$

If this condition is satisfied, then

$$(10.8)\text{b} \qquad w_{\omega} = \partial_{\omega}\Omega = \frac{1}{\sqrt{|\mathfrak{g}|}}\,\overset{-1}{k}_{\omega}{}^{\alpha}\,{}^{*}\overset{-1}{h}_{\alpha\beta}\partial_{\gamma}\sqrt{|\mathfrak{g}|}\,{}^{*}k^{\gamma\beta}.$$

PROOF. One obtains (10.8)b by multiplying (10.6) by $\overset{-1}{k}_{\omega}{}^{\mu}$. The conditions of integrability of (10.8)b are (10.8)a.

NOTE. Theorems (10.2) and (10.3) show under what circumstances *the field equation* (10.5) *could be enforced by conformal change* (9.1).

11. Conformally invariant connection $\tilde{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$.

LEMMA (11.1). *Put*

$$(11.1)\text{a} \qquad Z_{\omega\lambda\mu} \overset{\text{def}}{=} s_{\lambda\omega}{}^{\alpha}g_{\alpha\mu} + s_{\omega\mu}{}^{\alpha}g_{\lambda\alpha} + u^{\alpha}{}_{\lambda\omega}g_{\alpha\mu} + u^{\alpha}{}_{\omega\mu}g_{\lambda\alpha}.$$

Then

$$(11.1)\text{b} \qquad Z_{\omega\lambda\mu} = w_{\omega}g_{\lambda\mu}.$$

PROOF. The tensor $Z_{\omega\lambda\mu}$ may be written

$$Z_{\omega\lambda\mu} = 2s_{\omega[\mu\lambda]} + 2u_{(\mu\lambda)\omega} + 2s_{\omega(\mu}{}^{\alpha}k_{\lambda)\alpha} + 2u_{\alpha\omega[\mu}k_{\lambda]}{}^{\alpha}.$$

Substituting for the last two terms from (9.5) and using Lemma (9.1), we easily obtain (11.1)b.

We shall need this Lemma in the proof of the next Theorem. In order to state this Theorem clearly we have to explain some new notation. If F is a function of w_{λ}, then we denote by $\hat{F}$ the same function of the argument $\overset{-1}{k}_{\lambda}{}^{\alpha}S_{\alpha}$. Thus, for example,

$$\hat{w}_{\lambda} = \overset{-1}{k}_{\lambda}{}^{\alpha}S_{\alpha}.$$

We obviously have

$$'k_{\lambda}{}^{\nu} = k_{\lambda}{}^{\nu}, \quad '\overset{-1}{k}_{\lambda}{}^{\nu} = \overset{-1}{k}_{\lambda}{}^{\nu}$$

so that

$$'\hat{w}_{\lambda} = '\overset{-1}{k}_{\lambda}{}^{\alpha}\,'S_{\alpha} = \overset{-1}{k}_{\lambda}{}^{\alpha}\,'S_{\alpha}.$$

THEOREM (11.1). *Let $g_{\lambda\mu}$ be of the first class. Then there is only one connection $\tilde{\Gamma}_\lambda{}^\nu{}_\mu$ with the following properties:*

a) *it satisfies the conformally invariant conditions*

$$\partial_\omega g_{\lambda\mu} - \tilde{\Gamma}_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \tilde{\Gamma}_\omega{}^\alpha{}_\mu g_{\lambda\alpha} = \overset{-1}{k}_\omega{}^\alpha S_\alpha g_{\lambda\mu} \tag{11.2a}$$

$$\tilde{S}_\lambda \overset{\text{def}}{=} \tilde{\Gamma}_{[\lambda}{}^\alpha{}_{\alpha]} = 0, \tag{11.2b}$$

b) *it is invariant with respect to* (9.1)

$$'\tilde{\Gamma}_\lambda{}^\nu{}_\mu = \tilde{\Gamma}_\lambda{}^\nu{}_\mu. \tag{11.3}$$

This connection is given by

$$\begin{aligned}\tilde{\Gamma}_\lambda{}^\nu{}_\mu &= \Gamma_\lambda{}^\nu{}_\mu - \hat{s}_{\lambda\mu}{}^\nu - \hat{u}^\nu{}_{\lambda\mu} \\ &= \Gamma_\lambda{}^\nu{}_\mu - \tfrac{1}{2}[\hat{a}_{\lambda\mu}{}^\nu + \hat{l}_{\lambda\mu}{}^\nu + 3\hat{l}_{[\alpha\beta\gamma]} k_\lambda{}^\alpha k_\mu{}^\beta h^{\gamma\nu} - 2k_{(\lambda}{}^\alpha \hat{l}_{\mu)\gamma\alpha} h^{\gamma\nu}].\end{aligned} \tag{11.4}$$

PROOF. The equations (10.1) and (10.7) yield

$$w_\mu = ('S_\lambda - S_\lambda)\overset{-1}{k}_\mu{}^\lambda = '\hat{w}_\mu - \hat{w}_\mu. \tag{11.5}$$

The tensor $k_{\omega\mu\nu}$ defined by (9.10) is a homogeneous linear function of w_λ. Hence,

$$k_{\omega\mu\lambda} = e^{-\Omega}\,'\hat{k}_{\omega\mu\lambda} - \hat{k}_{\omega\mu\lambda} \tag{11.6}$$

where, of course,

$$'\hat{k}_{\omega\mu\lambda} = 2'\hat{w}_\alpha k_{[\omega}{}^\alpha\,'h_{\mu]\lambda} - '\hat{w}_\lambda\,'k_{\omega\mu}.$$

Therefore,

$$\begin{aligned}k_{\omega\mu\lambda} h^{\alpha\beta} &= e^{-\Omega} h^{\alpha\beta}\,'\hat{k}_{\omega\mu\lambda} - \hat{k}_{\omega\mu\lambda} h^{\alpha\beta} \\ &= 'h^{\alpha\beta}\,'\hat{k}_{\omega\mu\lambda} - h^{\alpha\beta}\hat{k}_{\omega\mu\lambda}.\end{aligned} \tag{11.7}$$

The tensor $Z^{\alpha\beta\gamma}_{\omega\mu\nu}$ involved in (9.11)a is a conformal invariant [1]. Therefore, there is an equation similar to (11.7) for $l_{\omega\mu\lambda}$, $'\hat{l}_{\omega\mu\lambda}$ and $\hat{l}_{\omega\mu\lambda}$. By the same token we obtain from (9.12) similar equations to

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[1] The scalars K and k involved in II (10.3)b are obviously conformal invariants. So are the tensors $\underset{x}{E}_\lambda{}^\nu$, since

$$\Sigma\, \underset{x}{E}_\lambda{}^\nu = \delta_\lambda^\nu.$$

The conformal invariance of $\overset{xy}{Q}_{\omega\mu}\overset{xy}{Q}{}^{\alpha\beta}$ is then evident.

(11.7) for $s_{\omega\mu\lambda}$, $'\hat{s}_{\omega\mu\lambda}$, $\hat{s}_{\omega\mu\lambda}$ and $u_{\nu\lambda\mu}$, $'\hat{u}_{\nu\lambda\mu}$, $\hat{u}_{\nu\lambda\mu}$ so that finally,

$$\begin{aligned} s_{\lambda\mu}{}^{\nu} &= {}'\hat{s}_{\lambda\mu}{}^{\nu} - \hat{s}_{\lambda\mu}{}^{\nu} \\ u^{\nu}{}_{\lambda\mu} &= {}'\hat{u}^{\nu}{}_{\lambda\mu} - \hat{u}^{\nu}{}_{\lambda\mu}.\,^{1} \end{aligned} \tag{11.8}$$

Substituting from (11.8) into (9.2)b we get, after a slight rearrangement,

$$\widetilde{\Gamma}_{\lambda}{}^{\nu}{}_{\mu} \overset{\text{def}}{=} \Gamma_{\lambda}{}^{\nu}{}_{\mu} - \hat{s}_{\lambda\mu}{}^{\nu} - \hat{u}^{\nu}{}_{\lambda\mu} = {}'\Gamma_{\lambda}{}^{\nu}{}_{\mu} - {}'\hat{s}_{\lambda\mu}{}^{\nu} - {}'\hat{u}^{\nu}{}_{\lambda\mu}. \tag{11.9}$$

The tensors $\hat{s}_{\lambda\mu}{}^{\nu}$ and $\hat{u}^{\nu}{}_{\lambda\mu}$ are to be constructed according to the pattern of (9.12).

$$\begin{aligned} 2\hat{s}_{\omega\mu\nu} &= \hat{l}_{\omega\mu\nu} + 3\hat{l}_{[\alpha\beta\nu]}k_{\omega}{}^{\alpha}k_{\mu}{}^{\beta} \\ 2\hat{u}_{\nu\lambda\mu} &= \hat{a}_{\lambda\mu\nu} - 2k_{(\lambda}{}^{\alpha}\hat{l}_{\mu)\nu\alpha}. \end{aligned} \tag{11.10}$$

In this way we obtain (11.4) from (11.8) and (11.9). If we remember II (2.1)a, we see from (11.4) that

$$\partial_{\omega}g_{\lambda\mu} - \widetilde{\Gamma}_{\lambda}{}^{\alpha}{}_{\omega}g_{\alpha\mu} - \widetilde{\Gamma}_{\omega}{}^{\alpha}{}_{\mu}g_{\lambda\alpha} = \hat{Z}_{\omega\lambda\mu}$$

where $\hat{Z}_{\omega\lambda\mu}$ is to be constructed in the usual way from $Z_{\omega\lambda\mu}$ given by (11.1)a. Therefore, (11.2)a holds by virtue of Lemma (11.1), which obviously is also valid for $\hat{Z}_{\omega\lambda\mu}$ and $\hat{w}_{\omega}$.

Using (11.4) we derive

$$\widetilde{S}_{\lambda\mu}{}^{\nu} = S_{\lambda\mu}{}^{\nu} - \hat{s}_{\lambda\mu}{}^{\nu}$$

so that

$$\widetilde{S}_{\lambda} = S_{\lambda} - \hat{s}_{\lambda}. \tag{11.11a}$$

On the other hand, we have from (10.3)b

$$\hat{s}_{\lambda} = \hat{w}_{\alpha}k_{\lambda}{}^{\alpha} = \overset{-1}{k}_{\alpha}{}^{\beta}S_{\beta}k_{\lambda}{}^{\alpha} = S_{\lambda}. \tag{11.11b}$$

The equation (11.2)b follows from (11.11)a,b.

Assume that there are two connections $\widetilde{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$ and $*\widetilde{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$ satisfying (11.2)a. Denote by $C_{\lambda\mu}{}^{\nu}$ the tensor

$$C_{\lambda\mu}{}^{\nu} \overset{\text{def}}{=} \widetilde{\Gamma}_{\lambda}{}^{\nu}{}_{\mu} - {}^{*}\widetilde{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}.$$

---

[1] Here, of course,

$$\begin{aligned} {}'\hat{s}_{\lambda\mu}{}^{\nu} &= {}'\hat{s}_{\lambda\mu\alpha}{}'h^{\alpha\nu} \\ {}'\hat{u}^{\nu}{}_{\lambda\mu} &= {}'\hat{u}_{\alpha\lambda\mu}{}'h^{\alpha\nu}. \end{aligned}$$

From (11.2)a we easily derive for $C_{\lambda\mu}{}^{\nu}$ the condition

$$(11.12) \qquad C_{\lambda\omega}{}^{\alpha} g_{\alpha\mu} + C_{\omega\mu}{}^{\alpha} g_{\lambda\alpha} = 0.$$

Comparing this system to II (2.1)a we see that the uniqueness of the solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ of II (2.1)a implies $C_{\lambda\mu}{}^{\nu} = 0$. Hence, there is only one solution $\tilde{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$ of (11.2).

An easy inspection of (11.2)a shows that

$$(11.13) \qquad \partial_\omega {}'g_{\lambda\mu} - \tilde{\Gamma}_{\lambda}{}^{\alpha}{}_{\omega} {}'g_{\alpha\mu} - \tilde{\Gamma}_{\omega}{}^{\alpha}{}_{\mu} {}'g_{\lambda\alpha} = (\overset{-1}{k}_{\omega}{}^{\alpha} S_\alpha + w_\omega) {}'g_{\lambda\mu}.$$

Using (11.5) we see that

$$\overset{-1}{k}_{\omega}{}^{\alpha} S_\alpha + w_\omega = \hat{w}_\omega + w_\omega = {}'\hat{w}_\omega = {}'\overset{-1}{k}_{\omega}{}^{\alpha} {}'S_\alpha.$$

Thus, (11.13) reduces to

$$(11.14) \qquad \partial_\omega {}'g_{\lambda\mu} - \tilde{\Gamma}_{\lambda}{}^{\alpha}{}_{\omega} {}'g_{\alpha\mu} - \tilde{\Gamma}_{\omega}{}^{\alpha}{}_{\mu} {}'g_{\lambda\alpha} = {}'\overset{-1}{k}_{\omega}{}^{\alpha} {}'S_\alpha {}'g_{\lambda\mu}.$$

Comparing (11.14) and (11.2)a we see that this latter equation is conformally invariant.

NOTE I. The equation (11.2)b may be looked upon as the field equation belonging to $\tilde{\Gamma}_{\lambda}{}^{\nu}{}_{\mu}$, which replaces the field equation (2.8)a belonging to $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$. While this latter equation has to be postulated, (11.2)b is satisfied automatically, even if (2.8)a does not hold.

NOTE II. The equations (11.2)a are substantially of the type suggested by Bose, S. N., (2) in his attempt to generalize Einstein's unified theory. [1]

This concludes the study of the geometrical aspect of field equations (2.8). The results obtained will lead us in a quite natural way to their physical interpretations, dealt with in the next chapter.

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[1] Another type of transformation of the connection is considered in Mishra, R. S., (1).

CHAPTER V

PHYSICAL INTERPRETATION OF FIELD EQUATIONS

PART I

Third class

1. Prerequisites [1]. In this chapter we shall consider the physical interpretations of the field equations

$$S_\lambda = 0, \tag*{(1.1)a}$$

$$\partial_{[\mu} X_{\lambda]} = R_{[\mu\lambda]}, \tag*{(1.1)b}$$

$$R_{(\mu\lambda)} = 0 \tag*{(1.1)c}$$

in four-dimensional space-time. The first part is concerned exclusively with the third class. The second part is devoted to the first two classes.

According to Definition I (1.1), the third class is characterized by

$$K = 0, \quad k = 0\,^{2} \tag*{(1.2)a}$$

so that I (3.3)b reduces to

$$g = \mathfrak{g}/\mathfrak{h} = 1. \tag*{(1.2)b}$$

Therefore, by virtue of II (7.1)c we have

$$U_\mu = U^{\alpha}{}_{\alpha\mu} = 0. \tag*{(1.2)c}$$

Equations I (15.2) reduce by virtue of (1.2) and I (2.11) to

$$f_{\lambda\nu} = -\frac{\sigma}{2}\, e_{\lambda\nu\alpha\beta} k^{\alpha\beta} = -\frac{\sigma}{2}\sqrt{|\mathfrak{h}|}\, \mathfrak{e}_{\lambda\nu\alpha\beta} k^{\alpha\beta} \tag*{(1.3)a}$$

$$f^{\lambda\nu} = \frac{\sigma}{2}\, E^{\lambda\nu\alpha\beta} k_{\alpha\beta} = \frac{\sigma}{2\sqrt{|\mathfrak{h}|}}\, \mathfrak{E}^{\lambda\nu\alpha\beta} k_{\alpha\beta} \tag*{(1.3)b}$$

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[1] This chapter is based on results in Hlavatý, V., (12), (13), (15), (16), (17), (20).

[2] *Cf.* I (1.5).

and yield

$$(1.4)\text{a} \qquad k^{\lambda\nu} = -\frac{\sigma}{2} E^{\lambda\nu\alpha\beta} f_{\alpha\beta} = -\frac{\sigma}{2\sqrt{|\mathfrak{h}|}} \mathfrak{E}^{\lambda\nu\alpha\beta} k_{\alpha\beta}$$ [1]

$$(1.4)\text{b} \qquad k_{\lambda\nu} = \frac{\sigma}{2} e_{\lambda\nu\alpha\beta} f^{\alpha\beta} = \frac{\sigma}{2} \sqrt{|\mathfrak{h}|}\, \mathrm{e}_{\lambda\nu\alpha\beta} f^{\alpha\beta}.$$

According to Theorem IV (4.1) the equation (1.1)a is equivalent to

$$(1.5)\text{a} \qquad \partial_{[\omega} f_{\mu\lambda]} = 0$$

which represents the integrability conditions of

$$(1.5)\text{b} \qquad f_{\mu\lambda} = \partial_{[\mu} f_{\lambda]}.$$ [2]

Substituting from (1.1)a, (1.2)b and I (3.9)b into II (6.14), one obtains

$$(1.5)\text{c} \qquad \partial_\lambda \sqrt{|\mathfrak{h}|}\, k^{\lambda\nu} = \sqrt{|\mathfrak{h}|}\, \nabla_\lambda k^{\lambda\nu} = 0.$$

This equation also follows from (1.3)a and (1.5)a.

In subsequent considerations we shall use the vector v^ν defined as follows

DEFINITION (1.1). *We put*

$$(1.6) \qquad v^\nu \sqrt{|\mathfrak{h}|} \overset{\text{def}}{=} \sigma \mathfrak{E}^{\nu\alpha\beta\gamma} S_{\alpha\beta\gamma}.$$

AGREEMENT (1.1). *In this part we assume*

$$(1.7) \qquad S_{[\alpha\beta\gamma]} S^{[\alpha\beta\gamma]} \geq 0.$$

The vector v^ν defined by (1.6) may be expressed in a slightly different way.

THEOREM (1.1). *We have*

$$(1.8) \qquad v^\nu \sqrt{|\mathfrak{h}|} = \frac{\sigma}{2} \mathfrak{E}^{\alpha\beta\gamma\nu} \partial_\alpha k_{\beta\gamma} = \frac{\sigma}{2} \mathfrak{E}^{\alpha\beta\gamma\nu} \nabla_\alpha k_{\beta\gamma}$$

$$= \frac{\sigma}{2} E^{\alpha\beta\gamma\nu} \sqrt{|\mathfrak{h}|}\, \partial_\alpha k_{\beta\gamma} = \frac{\sigma}{2} E^{\alpha\beta\gamma\nu} \sqrt{|\mathfrak{h}|}\, \nabla_\alpha k_{\beta\gamma}.$$

The proof follows from II (6.12), I (2.11)a and (1.6).

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[1] *Cf.* I (15.17).

[2] *Cf.* IV (4.2)b.

2. Tensors $K_{\omega\mu\nu}$, $S_{\omega\mu\nu}$, and $U_{\nu\omega\mu}$.

THEOREM (2.1). *The tensor* $K_{\omega\mu\nu}$ *defined by* II (3.5) *may be expressed in terms of* v^{ν} *and* $f_{\lambda\mu}$ *as follows:*

$$K_{\omega\mu\lambda} = \sigma(v^{\nu} e_{\omega\lambda\mu\nu} + e_{\omega\mu\alpha\beta} \nabla_{\lambda} f^{\alpha\beta}). \tag{2.1}$$

PROOF. By virtue of I (2.12)d the equation (1.8) yields:

$$v^{\nu} e_{\omega\lambda\mu\nu} = 3\sigma \nabla_{[\omega} k_{\lambda\mu]},$$

so that according to II (3.5)

$$K_{\omega\mu\lambda} = \sigma v^{\nu} e_{\omega\lambda\mu\nu} + 2 \nabla_{\lambda} k_{\omega\mu}.$$

Substituting into the last term of this equation from (1.4)b one obtains (2.1).

In the next Theorem we shall denote by $S^{\alpha\beta\gamma}_{\omega\mu\nu}$ the tensor $''Y^{\alpha\beta\gamma}_{\omega\mu\nu}$ defined by II(16.1)b and expressed in terms of $f_{\lambda\mu}$.

THEOREM (2.2). *Put*

$$2\, {}^{0}S_{\omega\mu\nu} \overset{\text{def}}{=} \sigma S^{\alpha\beta\gamma}_{\omega\mu\nu} e_{\alpha\beta\xi\eta} \nabla_{\gamma} f^{\xi\eta}. \tag{2.2)a}$$

Then

$$2S_{\omega\mu\nu} = \sigma v^{\delta} e_{\delta\alpha\beta\gamma} S^{\alpha\beta\gamma}_{\omega\mu\nu} + 2\, {}^{0}S_{\omega\mu\nu} \tag{2.2)b}$$

and the tensor ${}^{0}S_{\omega\mu\nu}$ *does not contain* v^{ν}.

PROOF. The equation II (17.3) is equivalent to

$$2\sigma S_{\omega\mu\nu} = \sigma K_{\alpha\beta\gamma} S^{\alpha\beta\gamma}_{\omega\mu\nu}.$$

Substituting into this equation from (2.1) we obtain (2.2)b. The statement about the tensor ${}^{0}S_{\omega\mu\nu}$ follows by simple inspection from (2.2)a, (1.8) and II (16.1)b.

In the next Theorem we shall denote by $U^{\alpha\beta\gamma}_{\nu\lambda\mu}$ the tensor $2S^{\alpha\beta\gamma}_{\nu(\lambda|\delta|} k_{\mu)}{}^{\delta}$ expressed in terms of $f_{\lambda\mu}$.

THEOREM (2.3). *Put*

$${}^{0}U_{\nu\lambda\mu} \overset{\text{def}}{=} \sigma\, {}^{0}S_{\nu(\lambda}{}^{\alpha} e_{\mu)\alpha\xi\eta} f^{\xi\eta}. \tag{2.3)a}$$

Then

$$U_{\nu\lambda\mu} = \sigma v^{\delta} e_{\delta\alpha\beta\gamma} U^{\alpha\beta\gamma}_{\nu\lambda\mu} + {}^{0}U_{\nu\lambda\mu}, \tag{2.3)b}$$

and the tensor ${}^{0}U_{\nu\lambda\mu}$ *does not contain* v^{ν}.

PROOF. The equation (2.3)b follows from (2.2)b and II (3.2). The statement about the tensor ${}^0U_{\nu\lambda\mu}$ is obvious.

Theorems (2.2) and (2.3) will play an important role in the physical interpretations of the field equations.

NOTE. Substituting the vector U_μ computed from (2.3)b into (1.2)c one finds

$$\sigma v^\delta e_{\delta\alpha\beta\gamma} U^{\alpha\beta\gamma}_{\nu\lambda\mu} h^{\nu\lambda} + {}^0U^\nu{}_{\nu\mu} = 0. \tag{2.4}$$

This equation does not restrict v^ν: it reduces to an identity by means of (1.6) and (1.2)a.

THEOREM (2.4). *The connection* $\Gamma_\lambda{}^\nu{}_\mu$ *defined by* II (2.1) *may be expressed in terms of* $h_{\lambda\mu}$, $f_{\lambda\mu}$ *and* v^ν *as follows:*

$$\begin{aligned} \Gamma_\lambda{}^\nu{}_\mu = \{\lambda{}^\nu{}_\mu\} + \sigma v^\delta e_{\delta\alpha\beta\gamma}(\tfrac{1}{2}S^{\alpha\beta\gamma}_{\lambda\mu\xi} + U^{\alpha\beta\gamma}_{\xi\lambda\mu})h^{\xi\nu} \\ + {}^0S_{\lambda\mu}{}^\nu + {}^0U^\nu{}_{\lambda\mu}. \end{aligned} \tag{2.5}$$

The proof follows from II (3.1), (2.2)b and (2.3)b.

3. Digression.

If we neglect the dielectric constant and the permeability and use suitable units, then Maxwell's equations for an electromagnetic field $f_{\lambda\mu}$ [1] acquire the following simple form:

$$\partial_{[\omega} f_{\mu\lambda]} = 0 \tag{3.1)a}$$

$$v^\nu \sqrt{|\mathfrak{h}|} = \partial_\lambda f^{\lambda\nu} \sqrt{|\mathfrak{h}|} = \sqrt{|\mathfrak{h}|}\, \nabla_\lambda f^{\lambda\nu}.^{1} \tag{3.1)b}$$

The equation of continuity of the charge and current, *i.e.*,

$$(\sqrt{|\mathfrak{h}|}\, \nabla_\lambda v^\lambda =)\ \partial_\lambda v^\lambda \sqrt{|\mathfrak{h}|} = 0, \tag{3.1)c}$$

follows from (3.1)b.

In the next Theorem we shall use the curvature tensors $H_{\omega\mu\lambda}{}^\nu$ and $H_{\mu\lambda}$ defined by IV (3.1) and IV (3.7)a.

THEOREM (3.1). *Every solution of* (3.1) *is also a solution of*

$$\Delta f_{\mu\lambda} = 2\nabla_{[\mu} v_{\lambda]} - G_{\mu\lambda\alpha\beta} f^{\alpha\beta} \tag{3.2)a}$$

where

$$G_{\mu\lambda\alpha\beta} \overset{\text{def}}{=} H_{\mu\lambda\alpha\beta} - 2H_{\beta[\mu} h_{\lambda]\alpha} \tag{3.2)b}$$

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[1] Here we use on purpose the symbols $f_{\lambda\mu}$, v^ν introduced in previous sections.

and

$$\Delta \overset{\text{def}}{=} h^{\alpha\beta} \nabla_\alpha \nabla_\beta. \tag{3.2)c}$$

PROOF. The equation (3.1)a is equivalent to

$$\nabla_\omega f_{\mu\lambda} + \nabla_\mu f_{\lambda\omega} + \nabla_\lambda f_{\omega\mu} = 0.$$

Applying $\nabla^\omega \overset{\text{def}}{=} h^{\lambda\omega}\nabla_\lambda$ to this equation we find

$$\Delta f_{\mu\lambda} + 2\nabla^\omega \nabla_{[\mu} f_{\lambda]\omega} = 0. \tag{3.3}$$

On the other hand, one obtains according to IV (3.3) and (3.1)b

$$\begin{aligned} 2\nabla^\alpha \nabla_{[\mu} f_{\lambda]\alpha} &= 2[\nabla_{[\mu} \nabla^\alpha f_{\lambda]\alpha} + H^\alpha{}_{[\mu\lambda]}{}^\beta f_{\beta\alpha} + H^\alpha{}_{[\mu|\alpha|}{}^\beta f_{\lambda]\beta}] \\ &= -2\nabla_{[\mu} v_{\lambda]} + 2(H_{\alpha[\mu\lambda]\beta} f^{\beta\alpha} - H_{\beta[\mu} h_{\lambda]\alpha} f^{\alpha\beta}). \end{aligned} \tag{3.4)a}$$

This equation can be simplified. We find according to IV (3.4)b

$$\begin{aligned} 2H_{\alpha[\mu\lambda]\beta} f^{\beta\alpha} &= (H_{\alpha\mu\lambda\beta} - H_{\alpha\lambda\mu\beta}) f^{\beta\alpha} = \\ &= (H_{\alpha\mu\lambda\beta} + H_{\lambda\alpha\mu\beta}) f^{\beta\alpha} = \\ &= -H_{\mu\lambda\alpha\beta} f^{\beta\alpha}, \end{aligned}$$

so that (3.4)a reduces to

$$2\nabla^\alpha \nabla_{[\mu} f_{\lambda]\alpha} = -2\nabla_{[\mu} v_{\lambda]} - G_{\mu\lambda\alpha\beta} f^{\beta\alpha}. \tag{3.4)b}$$

Substituting from (3.4)b into (3.3) we obtain (3.2)a.

NOTE I. The equations (3.2)a will be referred to as *condensed Maxwell's equations* for the field $f_{\lambda\mu}$.

NOTE II. It is not difficult to express v^ν by means of the vector f_λ involved in (1.5)b

$$2v^\nu = \Delta f^\nu - \nabla^\nu \nabla_\lambda f^\lambda + H_\alpha{}^\nu f^\alpha.\,^1$$

Substituting from this equation into (3.2)a, and using again (1.5)b, one obtains a set of differential equations for f_λ.

4. Field equations (1.1)b. In this section we shall use the tensor $*A_{[\omega\mu]}$ defined by IV (3.12)a and put

$$A_{\omega\mu} \overset{\text{def}}{=} *A_{[\omega\mu]} + S_{\omega\mu} \tag{4.1}$$

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[1] *Cf.* also Todeschini, B., (1).

where

$$S_{\omega\mu} \overset{\text{def}}{=} \partial_{[\omega} S_{\mu]} + S^{\beta}(\tfrac{1}{2}K_{\omega\mu\beta} - 2U_{\alpha\beta[\mu} k_{\omega]}{}^{\alpha}). \tag{4.2}$$

THEOREM (4.1). *Neither one of the conditions*

$$(1.1)\text{b,c } \textit{and } (1.2)\text{a} \tag{4.3}$$

yields

$$S_{\omega\mu} = 0. \tag{4.4)a}$$

PROOF. Substituting into (4.2) from II (17.3)[1] and II (18.3), one sees that (1.2)a cannot yield (4.4)a. Remembering that Theorem IV (3.2) asserts the equivalence of IV (3.13)a (of IV (3.13)b) and (1.1)b ((1.1)c), we see that neither one of the conditions (1.1)b,c could yield (4.4)a.

NOTE. The importance of this Theorem will be seen later. It is obvious that if and only if (4.4)a holds, we also have

$$A_{\omega\mu} = {}^{*}A_{[\omega\mu]}. \tag{4.4)b}$$

THEOREM (4.2). *The tensor* $^{*}A_{[\omega\mu]}$ *may be expressed in terms of* $f_{\lambda\mu}$ *and* v^{ν} *as follows:*

$$^{*}A_{[\omega\mu]} = {}^{0}A_{\omega\mu} + A_{\omega\mu\lambda}v^{\lambda} + A_{\omega\mu\lambda\nu}v^{\lambda}v^{\nu} + \tfrac{1}{2}a_{\omega\mu\lambda\nu}\nabla^{\lambda}v^{\nu}, \tag{4.5)a}$$

where

$$\tfrac{1}{2}a_{\omega\mu\lambda\nu} \overset{\text{def}}{=} e_{\nu\alpha\beta\gamma} U^{\alpha\beta\,\gamma}_{\delta\lambda[\omega} e_{\mu]\xi\eta\zeta} h^{\xi\delta} f^{\eta\zeta}, \tag{4.5)b}$$

and the tensors $^{0}A_{\omega\mu}$, $A_{\omega\mu\lambda}$, $A_{\omega\mu\lambda\nu}$ *have the following properties:*

a) *they are skew symmetric in* $\omega\mu$,

b) *they are well-defined functions of* $f_{\lambda\mu}$, $h_{\lambda\mu}$, *and their first two covariant derivatives,*

c) *they do not contain the arguments* $\Delta f_{\lambda\mu}$, v^{ν}, $\nabla^{\lambda}v^{\nu}$.

PROOF. Substitute into IV (3.12)a for $k_{\alpha\beta}$, U_{β}, $U_{\alpha\beta\gamma}$, $K_{\alpha\beta\gamma}$ from (1.4), (1.2)c, (2.3)b and (2.1) respectively. The Theorem then follows by simple inspection of the result of the above substitution.

THEOREM (4.3)a. *The system* (1.1)b *of field equations is equi-*

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[1] or from II (7.23).

valent to

$$\partial_{[\omega}X_{\mu]} = \tfrac{1}{2}(\nabla_\lambda v^\nu)\,(\sigma e_{\nu\omega\mu\lambda} + a_{\omega\mu\lambda\nu}) \tag{4.6a}$$
$$+ A_{\omega\mu\lambda}v^\lambda + A_{\omega\mu\lambda\nu}v^\lambda v^\nu + {}^0A_{\omega\mu} + \frac{\sigma}{2}e_{\omega\mu\alpha\beta}\Delta f^{\alpha\beta} + S_{\omega\mu}.$$

PROOF. We have by virtue of (2.1)

$$\tfrac{1}{2}\nabla^\lambda K_{\omega\mu\lambda} = \frac{\sigma}{2}[(\nabla_\lambda v^\nu)e_{\nu\omega\mu\lambda} + e_{\omega\mu\alpha\beta}\Delta f^{\alpha\beta}]. \tag{4.7}$$

Substituting from (4.7) and (4.5)a into IV (3.13)a, and remembering (4.2), we obtain (4.6)a.

THEOREM (4.3)b. *The system* (1.1)ab *of field equations is equivalent to* (3.1)a *and*

$$\partial_{[\omega}X_{\mu]} = \tfrac{1}{2}(\nabla_\lambda v^\nu)\,(\sigma e_{\nu\omega\mu\lambda} + a_{\omega\mu\lambda\nu}) \tag{4.6b}$$
$$+ A_{\omega\mu\lambda}v^\lambda + A_{\omega\mu\lambda\nu}v^\lambda v^\nu + {}^0A_{\omega\mu} + \frac{\sigma}{2}e_{\omega\mu\alpha\beta}\Delta f^{\alpha\beta}.$$

The proof based on Theorem (4.3)a and equation (4.2) is obvious.

NOTE. The system (4.6)b may be looked upon as a system of six partial differential equations for six unknowns $f_{\lambda\mu}$ expressed in terms of auxiliary variables

$$X_\lambda,\ h_{\lambda\mu},\ v^\nu \tag{*}$$

and their derivatives:

$$f_{\lambda\mu} = f_{\lambda\mu}(X_\cdot, h_{\cdot\cdot}, v^\cdot, \nabla_\cdot v^\cdot).^{1} \tag{4.8}$$

THEOREM (4.4). *Upon computing* $k_{\lambda\mu}$ *from* (1.4) *and* (4.8) *and substituting it into the defining equation* (1.6), *we obtain four partial differential equations* (3.1)b *of the second order for the four unknowns* v^ν.

PROOF. Equations (1.8) and (1.6) are obviously equivalent so that we may deal with (1.8). Substituting into its right-hand term for $k_{\beta\gamma}$, computed from (4.8) and (1.4), we obtain on the right hand side $\nabla_\lambda v^\nu$ as an argument, while the left hand term of (1.8) does not

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[1] It is not necessary for our purposes to single out other derivatives of the arguments (*).

contain it. Therefore, (1.8) is no longer an identity. It becomes a set of four partial differential equations for four unknowns v^ν. From (4.6)b we see that $f_{\lambda\mu}$ in (4.8) contains $\nabla^\lambda v^\nu$ but no other derivatives of v^ν. Therefore, the above set (1.8) is of the second order with respect to v^ν. Using (1.4) one finds that (1.8) is equivalent to (3.1)b.

NOTE I. Substituting into (4.8) the solution

(4.9)a $$v^\nu = v^\nu(X_\cdot, h_{\cdot\cdot})^{1}$$

of (3.1)b, we obtain $f_{\lambda\mu}$ expressed in terms of $h_{\lambda\mu}$ and X_λ

(4.9)b $$f_{\lambda\mu} = f_{\lambda\mu}(X_\cdot, h_{\cdot\cdot}).^{1}$$

NOTE II. There is still another path leading to (3.1)b. Substituting into (1.6) from II (6.12) and (1.4)b, we obtain

(4.10) $$v^\nu\sqrt{|\mathfrak{h}|} \overset{\text{def}}{=} \partial_\lambda f^{\lambda\nu}\sqrt{|\mathfrak{h}|}.$$

This equation does not have the operational meaning of Maxwell's equations. It is *a definition of* v^ν and not a set of differential equations for $f_{\lambda\mu}$. We could give it an operational meaning if we had still *another* definition of v^ν and substituted from it into (4.10). *The set* (4.6)b *could be looked upon as another definition of* v^ν. Substituting v^ν defined by (4.6)b into (4.10), we convert this latter defining equation into a *set* (3.1)b *of four partial differential equations for* $f_{\mu\lambda}$.

In order to derive some consequences from the previous results we need the following

LEMMA (4.1). *We have*

(4.11) $$\mathfrak{v}^2 \overset{\text{def}}{=} h_{\lambda\mu}(v^\lambda\sqrt{|\mathfrak{h}|})\,(v^\mu\sqrt{|\mathfrak{h}|}) \leq 0.^{2}$$

PROOF. From (1.8) and I (2.13)a we find

$$v_\nu\sqrt{|\mathfrak{h}|} = -\frac{\sigma}{2}\, e_{\xi\eta\zeta\nu}\sqrt{|\mathfrak{h}|}\,\nabla^\xi k^{\eta\zeta},$$

so that according to I (2.12)d, II (6.12) and (1.7)

$$\begin{aligned} v^\nu\sqrt{|\mathfrak{h}|}\,v_\nu\sqrt{|\mathfrak{h}|} &= -\tfrac{3}{2}|\mathfrak{h}|\,\nabla_{[\alpha}k_{\beta\gamma]}\nabla^{[\alpha}k^{\beta\gamma]} \\ &= -6|\mathfrak{h}|\,S_{[\alpha\beta\gamma]}S^{[\alpha\beta\gamma]} \leq 0. \end{aligned}$$

NOTE. We know that (1.1)a is equivalent to the first set (3.1)a.

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[1] *Cf.* the footnote on the preceding page.

[2] We remind the reader that the signature of $h_{\lambda\mu}$ is $(+++-)$.

Moreover, we obtained the second set (3.1)b via (4.6)b, equivalent to (1.1)a,b. *Thus,* (3.1)a,b *result as consequences of* (1.1)a,b. Moreover, (3.1)a,b have the form of Maxwell's equations. This fact justifies the following

AGREEMENT (4.1). *We identify*

a) $f_{\lambda\nu}$ *with the electromagnetic field tensor,*
b) $v^\nu\sqrt{|\mathfrak{h}|}$ *with the current density vector,*
c) (3.1)c *with the continuity equation of the charge and current,*
d) $\mathfrak{v} = \pm\, |\mathfrak{v}^2|^{\frac{1}{2}}$ *with the charge density whenever* $\mathfrak{v}^2 \neq 0.$ [1] [2]

5. Generalized Maxwell's set.

THEOREM (5.1). *A necessary and sufficient condition for pure radiation*

(5.1)a
$$v^\nu = 0$$

is

(5.1)b
$$S_{[\omega\mu\lambda]} = 0.$$

The proof follows at once from (1.6). [3]

NOTE. Each of the conditions (5.1)ab is equivalent to

(5.1)c
$$\partial_\lambda f^{\lambda\nu}\sqrt{|\mathfrak{h}|} = \sqrt{|\mathfrak{h}|}\,\nabla_\lambda f^{\lambda\nu} = 0.$$

THEOREM (5.2). *The system* (4.6)b *is equivalent to Maxwell's equations* (3.1) *in the sense that every solution of* (4.6)b *is also a solution of* (3.1).

The proof will be accomplished in four steps.

1) We know from Lichnerowicz' result (3) that (1.1) admits a unique solution with prescribed boundary conditions for $g_{\lambda\mu}$ along $x^{\text{IV}} = \text{const}$.

2) The system (4.6)a, equivalent to (1.1)b, might admit a so-

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[1] According to the previous Lemma we have $\mathfrak{v}^2 < 0$ whenever $\mathfrak{v}^2 \neq 0$.

[2] For another definition of current and charge in some special cases of $g_{\lambda\mu}$ see Mavrides, S., (5). Another identification in the general case is in Mavrides., S., (1), (2), (3), (4).

[3] Using the method of successive approximations, Schrödinger, E., (6) proved that (5.1)b is not a consequence of (1.1). On the other hand, there are tensors $g_{\lambda\mu}$ which satisfy (1.1) and (5.1)b (*Cf.* the first example in section IV (5)). The condition (5.1)b appears also in Stephenson, G., (3).

lution satisfying

$$\partial_{[\omega}f_{\mu\lambda]} \neq 0. \tag{5.2}$$

This holds even when (4.4)a is satisfied, in which case (4.6)a reduces to (4.6)b. On the other hand, we know (*Cf.* Theorem (4.1)) that, *in the frame of our conditions* (1.1)a and (4.3), it is only (1.1)a which implies (4.4)a. The condition (1.1)a is equivalent to (3.1)a so that (5.2) cannot hold.

3) Assume that we are dealing with a charged particle so that (5.1) is not satisfied. By the manner in which we obtained (3.1)b from the defining equation (1.6) by means of (4.6)b, it is clear that every solution of (4.6)b must satisfy (3.1)b.

4) If we are dealing with pure radiation, then (5.1)b must be satisfied. Then the conditions imposed on $g_{\lambda\mu}$ are (5.1)b and

(5.3) (1.1), (1.2)a,

and (4.6)b reduces to

$$\partial_{[\omega}X_{\mu]} = {}^0A_{\omega\mu} + \frac{\sigma}{2}\, e_{\omega\mu\alpha\beta}\Delta f^{\alpha\beta}. \tag{5.4}$$

In other words, we have

$$\begin{aligned} Y_{\omega\mu} \overset{\text{def}}{=} \tfrac{1}{2}(\nabla^\lambda v^\nu)\,(\sigma e_{\nu\omega\mu\lambda} + a_{\omega\mu\lambda\nu}) \\ + A_{\omega\mu\lambda}v^\lambda + A_{\omega\mu\lambda\nu}v^\lambda v^\nu = 0. \end{aligned} \tag{5.5}$$

The system (4.6)b might admit a solution with

$$\partial_\lambda \sqrt{|\mathfrak{h}|}\, f^{\lambda\nu} \neq 0 \tag{5.6}$$

even if (5.5) holds. We see from (1.6) that (5.5) cannot be a consequence of (1.2)a.[1] According to the Note following Theorem II (7.7), we know that (1.1)a for the third class is equivalent to II (7.25). This latter equation may be rewritten according to (2.3)b and (1.4)a as

$$v^\delta e_{\delta\alpha\beta\gamma} U^{\alpha\beta\gamma}_{\nu\lambda\mu} E^{\lambda\nu\xi\eta} f_{\xi\eta} + \sigma^0 U_{\nu\lambda\mu} E^{\lambda\nu\xi\eta} f_{\xi\eta} = 0.$$

It is not difficult to see that this condition, equivalent to (1.1)a, does not yield (5.5). The same is true for (1.1)b. Finally, the form IV (3.13)b of (1.1)c shows that even this condition cannot yield (5.5). By the same token the conditions (5.3) combined together do not yield (5.5). Hence, the only remaining condition *in the frame of our restrictions* (5.1)b and (5.3) which yields (5.5) is (5.1)b. This latter condition is equivalent to (5.1)c so that (5.6) does not hold.

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[1] An easy inspection based on II (18.3) shows that neither one of the scalars K, k nor their derivatives are involved in $\mathrm{R}_{[\mu\lambda]}$.

THEOREM (5.3). *The system* (4.6)b *is equivalent to the set* (3.2)a *of condensed Maxwell's equation in the sense that every solution of* (4.6)b *is also a solution of* (3.2)a.

The proof follows easily from Theorems (3.1) and (5.2).

Previous results justify the following

DEFINITION (5.1). *The system* (4.6)b *will be referred to as the generalized Maxwell set.*

NOTE I. The generalized Maxwell set may be reduced to a set of four equations not containing X_λ,

$$\nabla_{[\xi}\{\tfrac{1}{2}(a_{\omega\mu]\lambda\nu} - \sigma e_{\omega\mu]\lambda\nu})\nabla^\lambda v^\nu + A_{\omega\mu]\lambda} v^\lambda + A_{\omega\mu]\lambda\nu} v^\lambda v^\nu + {}^0A_{\omega\mu]} + \frac{\sigma}{2} e_{\omega\mu]\alpha\beta} \Delta f^{\alpha\beta}\} = 0. \tag{5.7}$$

These equations follow from (4.6)b and

$$\partial_{[\xi}\partial_\omega X_{\mu]} = 0.$$

In any case, the generalized Maxwell set is not linear in $f_{\lambda\mu}$ *and its derivatives.* This fact distinguishes it from Maxwell's set (3.1) as well as from the condensed Maxwell set (3.2)a.

NOTE II. Theorem (4.4), based on (4.6)b, holds also in a slightly different form for (5.7). The proof is similar to the corresponding proof of Theorem (4.4). It is not based on the solution (4.8) of (4.6)b but on the solution

$$f_\lambda = f_\lambda(h_{\cdot\cdot}, v^\cdot, \nabla_\cdot v^\cdot, \nabla_\cdot \nabla_\cdot v^\cdot) \tag{5.8}$$

of (5.7) and on (1.5)b, following from (3.1)a.

6. Auxiliary results. In order to find physical interpretations of (1.1)c we need some auxiliary Theorems which will be proved in this section.

In the first Theorem we shall express the tensor ${}^*A_{(\mu\lambda)}$, defined by IV (3.12)b, in terms of $f_{\lambda\mu}$ and v^ν.

THEOREM (6.1). *The tensor* ${}^*A_{(\mu\lambda)}$ *may be expressed in terms of* $f_{\lambda\mu}$ *and* v^ν *as follows:*

$$(6.1)\qquad {}^*A_{(\mu\lambda)} = {}^0B_{\mu\lambda} + B_{\mu\lambda\nu}v^\nu + B_{\mu\lambda\omega\nu}v^\omega v^\nu + \sigma(\nabla^\alpha v^\beta)e_{\beta\xi\eta\zeta}U^{\xi\eta\zeta}_{\alpha\lambda\mu}$$

where the tensors ${}^0B_{\mu\lambda}$, $B_{\mu\lambda\nu}$, $B_{\mu\lambda\nu\omega}$ *have the following properties:*

a) *they are symmetric in* $\mu\lambda$,

b) *they are well-defined functions of* $f_{\lambda\mu}$ *and their first two covariant derivatives,*

c) *they do not contain the argument* v^ν *or its derivatives.*

The proof is similar to the proof of Theorem (4.2). Of course, here we have to use IV (3.12)b instead of IV (3.12)a.

Another tensor used later is the tensor

$$(6.2)\qquad \tfrac{1}{2}Q_{\mu\lambda} \overset{\text{def}}{=} \tfrac{1}{4}K_{\mu\alpha}{}^{\beta}K_{\lambda\beta}{}^{\alpha} + {}^*A_{(\mu\lambda)}$$
$$- \tfrac{1}{8}(K_{\alpha\beta\gamma}K^{\alpha\gamma\beta} + 4\,{}^*A)h_{\lambda\mu} - \tfrac{1}{2}e_{\lambda\alpha\beta\gamma}e_{\mu\xi\eta\zeta}S^{\alpha\beta\gamma}S^{\xi\eta\zeta} = \tfrac{1}{2}Q_{\lambda\mu}$$
$$({}^*A = {}^*A_{\lambda\mu}h^{\lambda\mu}).$$

THEOREM (6.2). *The tensor* $Q_{\lambda\mu}$ *may be expressed in terms of* $f_{\lambda\mu}$ *and* v^ν *as follows:*

$$(6.3)\qquad \tfrac{1}{2}Q_{\mu\lambda} = \tfrac{1}{4}e_{\mu\alpha\beta\gamma}e_{\lambda\xi\eta\zeta}(\nabla^\xi f^{\beta\gamma})\,(\nabla^\alpha f^{\eta\zeta}) + {}^0B_{\mu\lambda}$$
$$+ v^\nu(B_{\mu\lambda\nu} + \nabla_{(\lambda}f_{\mu)\nu}) + v^\nu v^\omega B_{\mu\lambda\omega\nu} + \Lambda h_{\lambda\mu} + \sigma(\nabla^\alpha v^\beta)e_{\beta\xi\eta\zeta}U^{\xi\eta\zeta}_{\alpha\lambda\mu}$$

where

$$(6.4)\text{a}\qquad \Lambda \overset{\text{def}}{=} -\tfrac{3}{4}v^2 - \tfrac{1}{2}{}^*A + \tfrac{1}{4}(\nabla^\alpha f^{\beta\gamma})\,(\nabla_\alpha f_{\beta\gamma}),$$

and

$$(6.4)\text{b}\qquad v^2 \overset{\text{def}}{=} v^\lambda v_\lambda.$$

PROOF. From (2.1) we find, after some elementary computation,

$$(6.5)\text{a}\qquad \tfrac{1}{4}K_{\mu\alpha\beta}K_\lambda{}^{\beta\alpha} = \tfrac{1}{2}v_\lambda v_\mu - \tfrac{1}{2}h_{\lambda\mu}v^2 + v^\alpha\nabla_{(\lambda}f_{\mu)\alpha}$$
$$+ \tfrac{1}{4}e_{\mu\alpha\beta\gamma}e_{\lambda\xi\eta\zeta}(\nabla^\xi f^{\beta\gamma})\,(\nabla^\alpha f^{\eta\zeta}).$$

Consequently,

$$(6.5)\text{b}\qquad -\tfrac{1}{8}K_{\alpha\beta\gamma}K^{\alpha\gamma\beta} = -\tfrac{1}{4}v^2 + \tfrac{1}{4}(\nabla^\alpha f^{\beta\gamma})\,(\nabla_\alpha f_{\beta\gamma}).$$

On the other hand, we derive from (1.6)

$$(6.5)\text{c}\qquad -\tfrac{1}{2}e_{\lambda\alpha\beta\gamma}e_{\mu\xi\eta\zeta}S^{\alpha\beta\gamma}S^{\xi\eta\zeta} = -\tfrac{1}{2}v_\lambda v_\mu.$$

Substituting from (6.5)abc into (6.2), we obtain, by virtue of (6.1), the equation (6.3).

NOTE I. The defining equation (6.2) is purely geometrical. It expresses $Q_{\mu\lambda}$ in terms of

$$S_{\alpha\beta\gamma},\ U_{\alpha\beta\gamma},\ K_{\alpha\beta\gamma} \text{ and } g_{\alpha\beta}.$$

The same tensor may be expressed in terms of physical quantities $f_{\lambda\mu}$, v^{ν}, and auxiliary variables in the form (6.3). We shall need both equations (6.2) and (6.3) later.

NOTE II. The equation (6.4)a can be expressed in a slightly different way. The first equation (1.2)a may be rewritten by virtue of (1.4)

$$4K = k_{\alpha\beta}k^{\alpha\beta} = -f_{\alpha\beta}f^{\alpha\beta} = 0$$

so that

$$(6.6) \qquad (\nabla^{\alpha}f^{\beta\gamma})(\nabla_{\alpha}f_{\beta\gamma}) = -f^{\beta\gamma}\Delta f_{\beta\gamma}.$$

The second term in (6.6) can easily be computed from (4.6)b. In this way we obtain for the last term, on the right-hand side of (6.4)a, an explicit function of $\partial_{[\mu}X_{\lambda]}$. Therefore, the choice of X_{λ} affects the value of the scalar Λ.

7. Digression. In order to understand the results of the next section we have to remind the reader of some facts about the field equations IV (3.16) of the gravitational theory.[1] We know that the constant $^{*}k$ involved in IV (3.16) is a constant multiple of the gravitational constant. The momentum energy tensor $^{*}T_{\mu\lambda}$ of matter, for the simple case of incoherent matter, may be written

$$(7.1) \qquad {}^{*}T_{\mu\lambda} = w_{\mu}w_{\lambda}$$

where the vector field w_{μ} is tangential to the trajectories of the particles. Assuming that these trajectories are not minimal curves, we may express w_{μ} as follows:

$$(7.2) \qquad w_{\mu} = \sqrt{M}\,u_{\mu}.$$

Here $M\sqrt{|\mathfrak{h}|}$ is the mass density of the matter, and u_{λ} is a unit vector. Therefore, IV (3.16) acquire the form

$$(7.3) \qquad H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = {}^{*}kMu_{\lambda}u_{\mu}.$$

The equations IV (3.4)e and IV (3.16) lead to the law of conservation of momentum and energy

$$(7.4)\text{a} \qquad \nabla_{\lambda}{}^{*}T^{\lambda\nu} = 0$$

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[1] By gravitational theory we mean the theory usually called the general theory of relativity.

which in our case takes the form

$$0 = \nabla_\lambda M u^\lambda u^\nu = u^\nu \nabla_\lambda M u^\lambda + M u^\lambda \nabla_\lambda u^\nu. \tag{7.4)b}$$

The vector u^ν being a unit vector, the vector $u^\lambda \nabla_\lambda u^\nu$ is either equal to zero or perpendicular to u^ν. Therefore, (7.4)b splits into two equations

$$u^\lambda \nabla_\lambda u^\nu = 0, \tag{7.5)a}$$

$$\nabla_\lambda M u^\lambda = 0. \tag{7.5)b}$$

If we denote by s the arc length of the trajectory under consideration, we find easily that (7.5)a is equivalent to

$$\frac{d^2x^\nu}{ds^2} + \{\lambda{}^\nu{}_\mu\} \frac{dx^\lambda}{ds} \frac{dx^\mu}{ds} = 0. \tag{7.6}$$

This set of four equations defines an autoparallel line with respect to $\{\lambda{}^\nu{}_\mu\}$ which line is usually referred to either as *a geodesic* with respect to the gravitational field $h_{\lambda\mu}$ or as a *path* of $\{\lambda{}^\nu{}_\mu\}$. Therefore, the physical meaning of (7.6) may be expressed by *the gravitational law of inertia:* If a particle moves freely in the gravitational field $h_{\lambda\mu}$, it describes a geodesic of this field. Moreover, we know that under certain conditions the first three equations (7.6) reduce to the Newtonian gravitational law at an arbitrary point, while the fourth one reduces to an identity at this point.

The equation (7.5)b is obviously equivalent to

$$\partial_\lambda M \sqrt{|\mathfrak{h}|}\, u^\lambda = 0 \tag{7.7}$$

which is *the equation of continuity of matter*. Therefore, we see that *this equation is a consequence of the gravitational field equations.*[1]

NOTE. In cosmological theory the gravitational field equations are slightly different from IV (3.16), namely

$$H_{\lambda\mu} - \tfrac{1}{2} H h_{\lambda\mu} = {}^*k\, {}^*T_{\lambda\mu} + {}^*\Lambda\, h_{\lambda\mu}. \tag{7.8}$$

If we wish to preserve the conservation law (7.4)a, then we must have by virtue of (7.8) and IV (3.4)c

$$\nabla_\lambda {}^*\Lambda\, h^{\lambda\nu} = h^{\lambda\nu} \nabla_\nu {}^*\Lambda = 0,$$

---

[1] Note that the equation (3.1)c of continuity of the charge and current follows from (3.1)b which is a consequence of the generalized Maxwell set (4.6)b.

and this equation is equivalent to

$$\partial_\mu {}^*\Lambda = 0. \tag{7.9}$$

Hence, $^*\Lambda$ is a constant. It is referred to as *the cosmological constant.* It is assumed to be negligibly small and has negligible effect so far as distances within our solar system are concerned.

8. Gravitational field equations. Metric. In this section we shall use the results of the two previous sections. The expressions $H_{\lambda\mu}$ and H involved in the next Theorem are defined by IV (3.7)a and IV (3.8).

THEOREM (8.1). *The system* (1.1)a, (1.1)c *of field equations is equivalent to* (3.1)a *and*

$$H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = \tfrac{1}{2}T_{\lambda\mu}, \tag{8.1}$$

where

$$T_{\lambda\mu} \overset{\text{def}}{=} v_\lambda v_\mu + Q_{\lambda\mu} = T_{\mu\lambda}, \tag{8.2}$$

and the middle member of (8.2) *is defined by* (6.3) *and* (3.1)b.

PROOF. First we obtain from (1.1)a and IV (3.13)b

$$H_{\lambda\mu} = \tfrac{1}{4}K_{\mu\alpha}{}^{\beta}K_{\lambda\beta}{}^{\alpha} + {}^*\Lambda_{(\mu\lambda)}, \tag{8.3}$$

so that

$$H = \tfrac{1}{4}K_{\alpha\beta\gamma}K^{\alpha\gamma\beta} + {}^*\Lambda. \tag{8.4}$$

Consequently, we have according to (6.2) and (6.5)c

$$H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = \tfrac{1}{2}(Q_{\mu\lambda} + v_\mu v_\lambda),$$

and this equation is equivalent to (8.1) where $T_{\lambda\mu}$ is defined by (8.2).

NOTE. (8.1) has a form similar to IV (3.16). The tensor $T_{\mu\lambda}$ has a form similar to (7.1), but for the term $Q_{\mu\lambda}$ which is due to electromagnetism. These facts justify the following

AGREEMENT (8.1). *We identify $h_{\lambda\mu}$ with the tensor of the gravitational field and $T_{\lambda\mu}$ with the momentum energy tensor of matter and* *electromagnetism.*[1] *The equations* (8.1) *will be referred to as gravitational field equations.*

NOTE I. We mentioned in the previous section that the equation of continuity for $\left\{\begin{matrix}\text{matter}\\ \text{charge and current}\end{matrix}\right\}$ lies within the frame of the $\left\{\begin{matrix}\text{gravitational}\\ \text{unified}\end{matrix}\right\}$ theory. On the other hand, the equation of continuity for matter *cannot be derived from the field equations* (1.1)abc. If we *assume* its validity for our new theory as well, then we may derive a relationship between the mass density $M\sqrt{|\mathfrak{h}|}$ and electric charge density $\mathfrak{v} = \pm \sqrt{|v^2\mathfrak{h}|}$.[2]

If the trajectory of a charged particle is not a minimal one, and if M is the mass of the particle, then we have

$$\sqrt{|\mathfrak{h}|}\, v^\nu = \mathfrak{v} u^\nu = \varphi M \sqrt{|\mathfrak{h}|}\, u^\nu \tag{8.5}$$

where u^ν is the unit vector and φ a factor of proportionality. This equation, together with (3.1)c and (7.7), leads to

$$u^\nu \partial_\nu \varphi = 0,$$

so that the function of position φ, which is involved in the formula

$$\mathfrak{v} = \varphi M \sqrt{|\mathfrak{h}|},\text{[3]} \tag{8.6}$$

is constant along the trajectory of the particle. If we substitute from (8.6) into the middle member of (8.2), we find that $T_{\lambda\mu}$ has the following arguments

$$u^\nu,\ h_{\lambda\mu},\ f_{\lambda\mu},\ M,\ \varphi. \tag{8.7}$$

The first four quantities are arguments of the momentum energy tensor $^*T_{\lambda\mu}$ for matter and electromagnetism in the gravitational theory in the presence of electromagnetism. This is one of the reasons justifying the identification of $T_{\lambda\mu}$ with the momentum energy tensor for matter and electromagnetism. Note that $T_{\lambda\mu}$ and

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[1] The full justification of this name will be given in the Note following this Agreement and in section (10).

[2] See (6.4)b.

[3] This formula relating mass density $M\sqrt{|\mathfrak{h}|}$ and electric charge density has been derived from the assumption that v^ν is not a null vector. If we agree to apply it also to the case of a null vector, then we have $\varphi M = 0$ for a particle moving along a minimal line (photons).

$^{*}T_{\lambda\mu}$ have different structures although they have the same first four arguments (8.7).

NOTE II. In the gravitational theory the gravitational tensor field $h_{\lambda\mu}$ plays the role of metric tensor. In order to determine the role of $h_{\lambda\mu}$ in the unified field theory, we must first define the characteristic features of a metric tensor.

DEFINITION (8.1). *Let* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ *be an arbitrary connection and* D_{μ} *the symbol of the corresponding covariant derivative. Let* $t_{\lambda\mu}$ *be a tensor with the following properties:*

a) *it is symmetric,*

b) *its determinant does not vanish,*

c) *any system of differential equations*

$$V^{\lambda} D_{\lambda} V^{\nu} = 0 \tag{8.8}$$

admits a first quadratic integral

$$V^{\lambda} V^{\mu} t_{\lambda\mu} = \text{const.} \tag{8.9}$$

Then $t_{\lambda\mu}$ *is termed a metric tensor belonging to the connection* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ [1].

THEOREM (8.2). $h_{\lambda\mu}$ *is the metric tensor belonging to the following connections:*

a) *the connection* $\{_{\lambda}{}^{\nu}{}_{\mu}\}$,

b) *the connection* $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ *defined by* II (2.1),

c) *the connection* $\Gamma_{(\lambda}{}^{\nu}{}_{\mu)}$ *built up from the connection sub b).*

PROOF. Denote by D_{μ} the symbol of covariant derivative with respect to any one of the connections mentioned sub a) b) c). From (8.8) we obtain

$$V^{\nu}\partial_{\nu} V^{\lambda} V^{\mu} h_{\lambda\mu} = V^{\nu} D_{\nu} V^{\lambda} V^{\mu} h_{\lambda\mu} = V^{\nu} V^{\lambda} V^{\mu} D_{\nu} h_{\lambda\mu}.$$

The last term may be written for the three cases a) b) c) as follows:

$$V^{\nu} V^{\lambda} V^{\mu} D_{\nu} h_{\lambda\mu} = \begin{cases} V^{\nu} V^{\lambda} V^{\mu} \nabla_{\nu} h_{\lambda\mu} \\ 2V^{\nu} V^{\lambda} V^{\mu} S_{\nu(\mu}{}^{\alpha} g_{\lambda)\alpha}\ {}^{2} \\ V^{\nu} V^{\lambda} V^{\mu} U_{\nu\lambda\mu}.\ {}^{3} \end{cases}$$

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[1] Schrödinger, E., (3) calls $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ and $t_{\lambda\mu}$ "physically compatible." *Cf.* also Winogradski, J., (1).

[2] *Cf.* II (2.2)a.

[3] *Cf.* II (7.7).

The term on the right hand side vanishes because of $\left\{\begin{array}{l}\text{II (8)}\\ \text{II (3).}\\ \text{II (7.1)b}\end{array}\right.$
Therefore, in each of these three cases (8.8) admits a first quadratic integral (8.9). Moreover, $h_{\lambda\mu}$ is symmetric and $\mathfrak{h}<0$. Thus, all requirements of Definition (8.1) are fulfilled.

9. Recapitulation and Continuation. Let us recapitulate first the results obtained in previous sections. The field equation (1.1)a is equivalent to the first set of Maxwell's equations

(9.1)a $$\partial_{[\omega}f_{\mu\lambda]} = 0$$

which represent integrability conditions of

(9.2) $$f_{\mu\lambda} = \partial_{[\mu}f_{\lambda]}.$$

Therefore, the six unknowns $f_{\mu\lambda}$ may be expressed in terms of four unknowns f_{μ}. Keeping in mind the equations (9.2) we may dismiss (1.1)a. The field equations (1.1)b for $f_{\mu\lambda}$ given by (9.2) are equivalent to the generalized Maxwell set of six equations

(9.3) $$\partial_{[\omega}X_{\mu]} = \tfrac{1}{2}(\nabla^{\lambda}v^{\nu})\,(\sigma e_{\nu\omega\mu\lambda} + a_{\omega\mu\lambda\nu}) + A_{\omega\mu\lambda}v^{\lambda} + A_{\omega\mu\lambda\nu}v^{\lambda}v^{\nu} + {}^{0}A_{\omega\mu} + \frac{\sigma}{2}e_{\omega\mu\alpha\beta}\Delta f^{\alpha\beta}.$$

Substituting its solution

(9.4) $$f_{\mu\lambda} = f_{\mu\lambda}(X_{\cdot},\, h_{\cdot\cdot},\, v^{\cdot},\, \nabla_{\cdot}v^{\cdot})$$

into the equation (1.6) defining v^{ν}, we obtain a set

(9.1)b $$v^{\nu}\sqrt{|\mathfrak{h}|} = \partial_{\lambda}f^{\lambda\nu}\sqrt{|\mathfrak{h}|}$$

of four partial differential equations for the unknowns v^{ν}. Every solution of the generalized Maxwell set (9.3) also satisfies Maxwell's equations (9.1)ab (or (9.2) and (9.1)b). The equations (1.1)c reduce by virtue of (9.2) to ten gravitational field equations

(9.5) $$H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = \tfrac{1}{2}T_{\lambda\mu}$$

for ten unknowns $h_{\lambda\mu}$. The gravitational tensor field $h_{\lambda\mu}$ is also the metric tensor of our space-time. Altogether we have $6 + 4 + 10 = 20$ equations (9.3), (9.1)b, and (9.5) for $4 + 4 + 4 + 10 = 22$

unknowns

$$X_{\lambda},\ f_{\lambda},\ v^{\nu},\ h_{\lambda\mu}.$$

We know from Lichnerowicz' result (3) that the system of field equations admits exactly one solution with *a priori* given suitable boundary conditions at $x^{\mathrm{IV}} = \text{const.}$

After this brief recapitulation we may proceed further with our investigations.

THEOREM (9.1). *The gravitational field* $h_{\lambda\mu}$ *and the electromagnetic field* $f_{\lambda\mu}$ *are interrelated. The* $\left\{\begin{matrix}\textit{gravitational}\\ \textit{electromagnetic}\end{matrix}\right.$ *field generates* $\left\{\begin{matrix}\textit{electromagnetic}\\ \textit{gravitational}\end{matrix}\right.$ *field. These fields as well as the electric charge*[1] *spring from one basic conception, namely from the unified tensor field* $g_{\lambda\mu}$.

PROOF. The equation (4.9)b shows that $f_{\lambda\mu}$ is expressible in terms of $h_{\lambda\mu}$. The equations (9.1)b and (9.5) show that $h_{\lambda\mu}$ is expressible in terms of $f_{\lambda\mu}$. The remaining part of the Theorem is obvious.

NOTE. In the unified theory we have to assume the presence of the electromagnetic field, $f_{\lambda\mu} \neq 0$, for if we assume $f_{\lambda\mu} = 0$, we obtain not only $g_{\lambda\mu} = h_{\lambda\mu}$,[2] but also

$$H_{\lambda\mu} = 0.^{3} \tag{9.6}$$

This set is known to all students of gravitational theory as the set of gravitational field equations if matter is represented by singularities.[4] (Sometimes one refers to (9.6) as gravitational field equations in empty space.)

The fact that electromagnetic field has to be present in the unified

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[1] If we adopt the continuity equation for matter, then the mass density $M\sqrt{|\mathfrak{h}|}$ also has this property (see equation (8.6)).

[2] See I (15.17).

[3] These equations follow from $f_{\lambda\mu} = 0$, (9.1)b and (9.5).

[4] This is the starting assumption in Einstein, A. — Infeld, L. — Hoffmann, B., (1) where the authors display an ingenious method for computing by successive approximations the trajectories of singularities of matter. The equations (7.3), or more generally,

$$H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = {}^{*}k\,{}^{*}T_{\lambda\mu}$$

hold for continuously distributed matter.

theory where $g_{\lambda\mu} \neq g_{\mu\lambda}$ makes of it a kind of *primary field existing a priori* in the frame of the unified field theory. Then we may say according to Theorem (9.1) *that gravitation*[1] *is generated by electromagnetism.* The question arises whether every electromagnetic field generates gravitation. The answer to this question is given by the next Theorem.

THEOREM (9.2). *There are electromagnetic fields which do not generate gravitation. In particular, the electromagnetic field of a plane wave in the electromagnetic theory of light does not generate gravitation.*[2]

PROOF. From the first example in section IV (5) we see that

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \quad X_\lambda = \partial_\lambda X$$

is a solution of (1.1)a,b,c provided $h_{\lambda\mu}$ is defined by IV (5.1) and $k_{\lambda\mu}$ is defined by IV (5.3). This latter set is by virtue of (1.3) equivalent to

$$f_{\text{I II}} = f_{\text{III IV}} = 0, \quad f_{\text{II III}} = f_{\text{IV II}} = -\sigma\cos\alpha, \qquad (9.7)$$
$$f_{\text{I IV}} = f_{\text{III I}} = \sigma\sin\alpha.$$

From IV (5.5)a we find that according to Theorem (5.1) we are dealing here with pure radiation[3]

$$\partial_\lambda f^{\lambda\nu}\sqrt{|\mathfrak{h}|} = 0.$$

The field (9.7) is known to be a field of a plane wave in the electromagnetic theory of light. The vector f_λ which defines $f_{\mu\lambda}$ by means of (9.2) is given here by

$$f_\lambda\left(2\sigma\int\sin\alpha\,dz,\ 2\sigma\int\cos\alpha\,dz,\ 0,\ 0\right)$$
$$(z \overset{\text{def}}{=} x^{\text{III}} - x^{\text{IV}}).$$

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[1] and mass density, if we adopt the continuity equation for matter.

[2] Hlavaty, V., (12), (13), (20). See also Takeno, H., (1).

[3] The same result could be obtained from the set

$$f^{\text{I II}} = f^{\text{III IV}} = 0, \quad f^{\text{II III}} = f^{\text{II IV}} = -\sigma\cos\alpha, \quad f^{\text{III I}} = f^{\text{IV I}} = \sigma\sin\alpha$$

which follows from (9.7) and

$$((h^{\lambda\nu})) = \left(\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right).$$

Of course, any other vector $f_\lambda + \partial_\lambda f$ with arbitrary f leads to the same field (9.7). From IV (5.1) we see that the curvature tensor $H_{\omega\mu\lambda}{}^{\nu}$ of $h_{\lambda\mu}$ vanishes. Thus, there is no gravitational field and $h_{\lambda\mu}$ represents only the metric tensor.

10. Momentum energy tensor $T_{\lambda\mu}$.

THEOREM (10.1). *The momentum energy tensor* $T_{\lambda\mu}$, *involved in the gravitational field equations* (8.1) *satisfies the conditions*

$$\nabla_\lambda T^{\lambda\nu} = 0. \tag{10.1}$$

(*Law of conservation of momentum and energy.*)

PROOF. The equation (10.1) follows easily from (8.1) and IV (3.4)e.

NOTE. Although the structure of $T_{\lambda\mu}$ is quite different from that of the momentum energy tensor in the gravitational theory, the equation (10.1) can be rewritten in a form familiar to all students of relativity.

THEOREM (10.2). *Put*

$$\begin{aligned}\mathscr{H} &= h^{\mu\lambda}\sqrt{|\mathfrak{h}|}[\{\lambda{}^{\alpha}{}_{\mu}\}\{\alpha{}^{\beta}{}_{\beta}\} - \{\beta{}^{\alpha}{}_{\mu}\}\{\alpha{}^{\beta}{}_{\lambda}\}] \\ &= -\sqrt{|\mathfrak{h}|}H + h^{\mu\lambda}\sqrt{|\mathfrak{h}|}(\partial_\mu\{\lambda{}^{\alpha}{}_{\alpha}\} - \partial_\alpha\{\lambda{}^{\alpha}{}_{\mu}\})\end{aligned} \tag{10.2}$$ [1]

$$\mathfrak{T}_\lambda{}^{\nu} \overset{\text{def}}{=} T_\lambda{}^{\nu}\sqrt{|\mathfrak{h}|} \tag{10.3a}$$

$$\mathfrak{t}_\lambda{}^{\nu} \overset{\text{def}}{=} \left(\delta_\lambda^\nu \mathscr{H} - \frac{\partial \mathscr{H}}{\partial \partial_\nu h^{\alpha\beta}}\,\partial_\lambda h^{\alpha\beta}\right). \tag{10.3b}$$ [2]

Then (10.1) *is equivalent to*

$$\partial_\nu(\mathfrak{T}_\lambda{}^{\nu} + \mathfrak{t}_\lambda{}^{\nu}) = 0. \tag{10.4}$$

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[1] $\mathscr{H}$ is not a scalar. There are coordinate systems for which $\mathscr{H}$ vanishes at an arbitrary point P. Such a system is, for example, one for which

$$\{\lambda{}^{\nu}{}_{\mu}\} = 0 \text{ at } P$$

so that according to (10.2)

$$\mathscr{H} = 0 \text{ at } P.$$

[2] $\mathfrak{t}_\lambda{}^{\nu}$ is not a tensor density. We use the gothic type only for historical reasons.

PROOF. We find from (9.5), (10.1) and II (7)

$$0 = \nabla_\nu \mathfrak{T}_\lambda{}^\nu = \partial_\nu \mathfrak{T}_\lambda{}^\nu - \{{}_{\lambda}{}^{\alpha}{}_{\nu}\}\mathfrak{T}_\alpha{}^\nu =$$
$$\partial_\nu \mathfrak{T}_\lambda{}^\nu - (\partial_\lambda h_{\beta\nu})\,(H^{\beta\nu} - \tfrac{1}{2}Hh^{\beta\nu})\sqrt{|\mathfrak{h}|}.$$

The proof that the last term on the right side of this equation is equal to $\partial_\nu \mathfrak{t}_\lambda{}^\nu$ may be derived either by straightforward computation based on (10.2) and (10.3) or found in any detailed book on gravitational theory. [1] Therefore, the last equation reduces to (10.4).

NOTE. The proof of Theorem (10.2) does not require the knowledge of the structure of $T_{\lambda\mu}$. [2]

NOTE. There is another interesting feature displayed by the tensor $T_{\lambda\mu}$. We see from (8.2) and (6.3) that $T_{\lambda\mu}$ contains a term $\Lambda h_{\lambda\mu}$ so that (9.5) could be written

$$(10.5) \qquad H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = \tfrac{1}{2}\tilde{T}_{\lambda\mu} + \Lambda h_{\lambda\mu}$$

where

$$\tfrac{1}{2}\tilde{T}_{\lambda\mu} = \tfrac{1}{2}T_{\lambda\mu} - \Lambda h_{\lambda\mu}.$$

Comparing (10.5) with (7.8), we see that Λ may be looked upon as *a cosmological function* [3], a generalization of the cosmological constant $^*\Lambda$ involved in (7.8). Note that in contradistinction to the gravitational theory $\Lambda h_{\lambda\mu}$ *is an argument of the momentum energy tensor* $T_{\lambda\mu}$, which yields law (10.1) of conservation of momentum and energy.

THEOREM (10.3). *The equations of motion are*

$$(10.6) \qquad v^\lambda \nabla_\lambda v^\nu + \nabla_\lambda Q^{\lambda\nu} = 0.$$

The proof follows easily from (3.1)c, (8.2), (9.1)b and (10.1).

NOTE. Equations (4.9)a express v^ν in terms of $h_{\lambda\mu}$ which are defined by ten gravitational field equations (8.1). The interdependence of v^ν and $h_{\lambda\mu}$ symbolized by (4.9)a is explicitly described by (10.6).

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[1] See for instance Bergmann, P., (1) p. 196.

[2] *Cf.* Treder, H. J., (1).

[3] The name cosmological function Λ refers here rather to the geometrical structure of the term $\Lambda h_{\lambda\mu}$ than to the physical meaning suggested by the cosmological constant $^*\Lambda$ which appears in the term $^*\Lambda h_{\lambda\mu}$ involved in the field equations of the gravitational theory. Thus, for instance, the scalar Λ could contain information about the pressure, about $^*\Lambda$ and about other scalar physical quantities.

11. Discussion of (10.6). The result of Theorem (10.3) is rather disappointing. We list below the three main objections which could be raised against (10.6).

1) The equation (10.6) does not show the role of the unified connection $\Gamma_\lambda{}^\nu{}_\mu$ (*Cf.* II (2.1)) with which we began. The equations of motion are expressed in terms of Christoffel symbols $\{_\lambda{}^\nu{}_\mu\}$ only.

2) The gravitational theory replaced the gravitational force vector by the gravitational field: the equations of a trajectory of a particle moving freely in the gravitational field may be brought into the form (7.6), which not only expresses the "gravitational law of inertia", but also yields, in the first approximation, the gravitational law of Newton. In the unified theory, where instead of the simple gravitational field $h_{\lambda\mu}$ we have the unified field

$$g_{\lambda\mu} = h_{\lambda\mu} + \frac{\sigma}{2} e_{\lambda\nu\alpha\beta} f^{\alpha\beta}, \tag{11.1}$$

the equation (10.6) replaces the gravitational force vector by the gravitational field $h_{\lambda\mu}$ but introduces, *in addition to the electromagnetic field*, a kind of "electromagnetic force vector" $\nabla_\lambda Q^{\lambda\nu}$, so that from this point of view (10.6) does not represent the unified law of inertia but rather a generalization of the second law of Newton.

3) The third objection concerns the application to celestial mechanics. Assuming that the electromagnetic field (of the sun) is so weak that quadratic (and higher) products of its components may be neglected, we obtain from (9.5) and (9.1)b the equations (9.6) which represent the gravitational field equations of the gravitational theory for empty space, whose Schwarzschild solution $h_{\lambda\mu}$ is known. Substituting it into (10.6) [1] we obtain a system different from (7.6). In other words, the predictions (based on (10.6)) concerning the three famous phenomena of celestial mechanics [2] will be at variance with the predictions obtained in the gravitational

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[1] And neglecting only the cubic (and higher) products of $f_{\lambda\mu}$.

[2] Namely the advance of the perihelion of Mercury, the deflection of light in a Schwarzschild gravitational field, and the shift of spectral lines.

theory (although both theories use the same Schwarzschild field $h_{\lambda\mu}$). This discrepancy between these two predictions constitutes a serious objection to the present theory.

In order to avoid these objections we must change some of the basic conditions in (1.1).

12. Another approach. As a guiding principle for this change we assume that the predictions of the above-mentioned three phenomena by means of gravitational theory have been confirmed within reasonable limits by experimental tests. Therefore, rather than impose some conditions directly on $g_{\lambda\mu}$, we shall try to formulate some general conditions in such a way that, in the special case of celestial mechanics, they will yield predictions confirming experimental tests which have been regarded as proof of the gravitational theory.

From this point of view, it seems reasonable to maintain the conditions (1.1)a and (1.1)c. The latter leads to the gravitational field equations as in the gravitational theory; the former is equivalent to the first set of Maxwell's equations. Therefore, our desired change may affect only (1.1)b. It is irrelevent whether we consider (1.1)b or its integrability conditions,

(12.1)
$$\partial_{[\omega}R_{\mu\lambda]} = 0.$$

(*Cf.* Note I following Definition (5.1).) In subsequent considerations we shall replace (12.1) by another set of four equations.

AGREEMENT (12.1). *Henceforth in this part, we replace* (1.1)b *equivalent to* (12.1) *by*

(12.2)a
$$D_\lambda G Q^{\lambda\nu} = 2U^{\nu}{}_{\alpha\lambda}H^{\alpha\lambda} + S_\alpha G Q^{\alpha\nu} + U_\alpha(GQ^{\alpha\nu} + 2Hh^{\alpha\nu})$$ [1]

where

(12.3)
$$G \overset{\text{def}}{=} 1/e = 1/(1 + \tfrac{1}{2}f_{\alpha\beta}f^{\alpha\beta} + f)$$ [2]

and D_λ is the symbol of covariant derivative with respect to the connection $\Gamma_\lambda{}^\nu{}_\mu$ defined by II (2.1).

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[1] If dealing with pure radiation (*Cf.* Theorem (5.1)), we do not have any equations of motion, although (10.1) holds. But (12.2)a could serve as a field equation even in the case of pure radiation.

[2] See I (15.14) and I (15.3).

NOTE I. We formulated this new field equation for the most general case. It is obvious that it reduces for the third class to

$$(12.2)\text{b} \qquad D_\lambda Q^{\lambda\nu} = 2U^\nu{}_{\alpha\lambda}H^{\alpha\lambda} + S_\alpha Q^{\alpha\nu}$$

so that by virtue of (1.1)a we have for the third class

$$(12.2)\text{c} \qquad D_\lambda Q^{\lambda\nu} = 2U^\nu{}_{\alpha\lambda}H^{\alpha\lambda}.$$

NOTE II. The field equations (1.1)a, (1.1)c may be looked upon as purely geometrical restrictions. According to (6.2) the same is true for (12.2)c. On the other hand, the equations (6.3), (2.3)b and (2.5) show that (12.2)c may be expressed in terms of $h_{\lambda\mu}$, $f_{\lambda\mu}$ and v^ν and we shall take these to have the same physical meaning as before.

It is now necessary to revise some previous results from the point of view of the new field equations. The results of the first three sections remain unchanged. Theorems (4.1)–(4.3) are in no relation to (12.2)c and have no significance here. According to the last part of the preceding Note II, the equation (4.8) [1] has to be replaced by a similar equation describing a solution of (12.2)c:

$$(12.4) \qquad f_{\lambda\mu} = f_{\lambda\mu}(h_{\cdot\cdot}, v^\cdot, \nabla_\cdot v^\cdot).$$

Using (12.4) we obtain (3.1)b in a way similar to that used in Theorem (4.4). Lemma (4.1), Agreement (4.1) and Theorem (5.1) remain unchanged. In Theorem (5.2) we have to replace (4.6)b by (12.2)c. The proof of this changed Theorem is based on the relation

$$(12.5) \qquad S_\alpha Q^{\alpha\nu} = 0$$

which reduces (12.2)b to (12.2)c. It is not difficult to prove that in the frame of our requirements (1.1)a, (1.1)c, (1.2)a and (12.2)c only the first one reduces (12.5) to an identity. [2] In Theorem (5.3) we have to perform the same change as in Theorem (5.2). Then, of course, (12.2)c will be referred to *as the generalized Maxwell's set* (*Cf.* Definition (5.1)).

Further changes will be discussed in the next section.

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[1] This equation describes a solution of (4.6)b.

[2] According to (6.2) the first term of (12.5) contains the vector

$$\tfrac{1}{4}S_\alpha K^{\alpha\beta\gamma}K^\nu{}_{\gamma\beta}.$$

Using II (6.14) one finds that the only restriction (in our frame of requirements) which annihilates this vector is (1.1)a. Moreover, we see from (6.2) that this vector appears only once in the first term of (12.5).

13. Unified law of inertia. The results of section 6 remain unchanged but for Note II which has to be omitted. Section 7 deals with gravitational theory, which is not affected by our change of field equations. Section 8 is based on (1.1)a and (1.1)c so that its results are not changed. Theorem (9.1) remains unchanged although its proof has to be slightly altered due to our change of one set of field equations. Theorem (9.2) holds even here: The fields IV (5.1) and (9.7), whose existence constituted the proof of Theorem (9.2), also satisfy (12.2)c.[1] Theorems (10.1) and (10.2) are obviously fulfilled, for they are independent of (1.1)b which we replaced by (12.2)c. The first substantial change to be enacted concerns Theorem (10.3) although (10.6) holds even here.

THEOREM (13.1). *The equations of motion are*

$$v^\lambda D_\lambda v^\nu = 0. \tag{13.1)a}$$

They are equivalent to

$$\frac{d^2x^\nu}{dq^2} + \Gamma_\lambda{}^\nu{}_\mu \frac{dx^\lambda}{dq}\frac{dx^\mu}{dq} = 0. \tag{13.1)b}$$

where

$$v^\nu = \frac{dx^\nu}{dq}. \tag{13.2}$$

PROOF. The equivalence of (13.1)a and (13.1)b is obvious. In order to prove (13.1)a we have to use II (7.1). According to these equations we have by virtue of (1.2)c

$$6U_{(\nu\lambda\mu)}h^{\lambda\mu} = 2U_{\nu\lambda\mu}h^{\lambda\mu} = 0. \tag{13.3}$$

Moreover, according to (1.1)a, (1.2)c and II (3.1)

$$D_\lambda Q^{\lambda\nu} = \nabla_\lambda Q^{\lambda\nu} + U^\nu{}_{\lambda\mu}Q^{\lambda\mu}. \tag{13.4)a}$$

In a similar way we find

$$v^\lambda D_\lambda v^\nu = v^\lambda \nabla_\lambda v^\nu + U^\nu{}_{\lambda\mu}v^\lambda v^\mu \tag{13.4)b}$$

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[1] From the equations following IV (5.8) we easily obtain by virtue of IV (5.5)a, IV (5.7), IV (3.12)b and (6.2)

$$Q_{\lambda\mu} = 0.$$

This equation together with IV (5.7) shows that (12.2)c is satisfied.

so that by virtue of (8.1), (8.2) and (13.3)

$$v^\lambda D_\lambda v^\nu + D_\lambda Q^{\lambda\nu} = v^\lambda \nabla_\lambda v^\nu + \nabla_\lambda Q^{\lambda\nu} + 2U^\nu{}_{\lambda\mu}(H^{\lambda\mu} - \tfrac{1}{2}Hh^{\lambda\mu})$$
$$= v^\lambda \nabla_\lambda v^\nu + \nabla_\lambda Q^{\lambda\nu} + 2U^\nu{}_{\lambda\mu}H^{\lambda\mu}.$$

This equation reduces by virtue of (10.6) and (12.2)c to (13.1)a.

THEOREM (13.2). (*The unified law of inertia*). *A charged particle moving freely in the unified field* (11.1) *describes a path* (*an autoparallel line*) *of the unified connection* $\Gamma_\lambda{}^\nu{}_\mu$ *given by* II (2.1).

PROOF. This Theorem describes in words the equations (13.1).

NOTE. Theorems (13.1) and (13.2) dispose of the first two objections discussed in Section 11. Therefore, we have to deal only with the third objection of that section. Neglecting the quadratic and higher products of $f_{\lambda\mu}$, we find that (12.2)c is automatically satisfied. Again neglecting the quadratic and higher products of $f_{\lambda\mu}$, we obtain, as before, the equation (9.6) with the Schwarzschild solution $h_{\lambda\mu}$ of the gravitational theory. Now we have to substitute this solution into (13.1)b, which may be written

$$\text{(13.5)} \qquad \frac{d^2x^\nu}{dq^2} + \{\lambda{}^\nu{}_\mu\}\frac{dx^\lambda}{dq}\frac{dx^\mu}{dq} + U^\nu{}_{\lambda\mu}v^\lambda v^\mu = 0.$$

In this equation we have to neglect only cubic and higher products of $f_{\lambda\mu}$. Now $U^\nu{}_{\lambda\mu}$ is, according to II (18.3), at least of the same order of magnitude as a quadratic product of $f_{\lambda\mu}$, and therefore [1], (13.5) reduces in our approximation to (7.6) *which is the equation of motion in the gravitational theory. Hence, our theory must yield the same predictions of the three aforementioned phenomena as does the gravitational theory. This result disposes of the third objection.*

By means of the unified law of inertia we replaced gravitational force vectors as well as electromagnetic force vectors by the unified field $g_{\lambda\mu}$. Now we have to show that this unified law of inertia may be brought into a form which expresses — at least at a point — the corresponding Newtonian laws. Only then could this theory be looked upon as a valid generalization of classical results. We shall deal with this problem in the next two sections.

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[1] By virtue of (3.1)b.

14. Auxiliary results. [1] In this section we deal with the tensor $g_{\lambda\mu}$ and the vector v^ν which are involved in (13.1). Let us denote by O an arbitrary point of our space-time whose coordinates we assume to be $x^\nu = 0$. Moreover, let us assume that $g_{\lambda\mu}$ may be expanded in a Taylor series in the neighborhood of O:

$$(14.1)\text{a} \quad g_{\lambda\mu} = (g_{\lambda\mu})_O + x^\alpha(\partial_\alpha g_{\lambda\mu})_O + \tfrac{1}{2}x^\alpha x^\beta(\partial_\alpha\partial_\beta g_{\lambda\mu})_O + \cdots$$

for sufficiently small x's. In subsequent considerations we shall use the equality sign $\overset{0}{=}$ ($\overset{1}{=}$) whenever we wish to stress that an equation holds only if we neglect linear (quadratic) and higher products of x's. [2] In particular, we assume

$$(14.1)\text{b} \qquad k_{\lambda\mu} = g_{[\lambda\mu]} \overset{0}{=} 0$$

so that

$$(14.1)\text{c} \qquad {}^0h_{\lambda\mu} \overset{\text{def}}{=} (h_{\lambda\mu})_O \overset{0}{=} g_{\lambda\mu}.\ ^3$$

These relations hold for any coordinate system. As far as the vector v^ν is concerned, we know from Theorem (8.2) that (13.1) admits a first integral

$$(14.2) \qquad h_{\lambda\mu}v^\lambda v^\mu = -\gamma^2 (= \text{const.}).$$

We shall confine ourselves to the case $\gamma^2 \neq 0$. Thus, according to the results of section 4, we have $\gamma^2 > 0$.

LEMMA (14.1). *There is always a coordinate system for which*

$$(14.3)\text{a} \qquad v^\nu \overset{0}{=} \gamma\delta^\nu_4, \quad (\gamma \overset{\text{def}}{=} \sqrt{\gamma^2}\,)$$

$$(14.3)\text{b} \qquad ((h_{\lambda\mu})) \overset{0}{=} \left(\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right).$$

PROOF. The tangential space of our space-time at O has as its ideal space P_3 a projective three dimensional space. The components ${}^0h_{\lambda\mu}$ may be regarded as projective homogeneous coordinates of a quadric Q in P_3. This quadric is a non-ruled one, because $h_{\lambda\mu}$ is of signature $(+ + + -)$. Moreover, the components $(v^\nu)_O$ may be looked upon as projective homogeneous coordinates of a point in P_3. This point is inside Q, since $\gamma^2 > 0$. If we refer Q to a coordinate system based on a self-polar tetrahedron, then we can easily obtain (14.3)b. If we take a self-polar tetrahedron whose vertex inside Q is the point v^ν, then we obtain (14.3)b and

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[1] In Sections 14 and 15 the Greek (Roman) indices run from 1 to 4 (from 1 to 3).

[2] The symbol $\overset{0}{=}$ is clearly equivalent to evaluation at the point O.

[3] *Cf.* the last Note in section 15. The conditions (14.1) imply that O is a privileged point as far as $g_{\lambda\mu}$ is concerned.

$$v^\nu \overset{0}{=} c\delta^\nu_4.$$

Comparing this equation with (14.2) we obtain $\gamma^2 = c^2$. Hence, a suitable orientation of the x^4 coordinate yields (14.3)a.

DEFINITION (14.1). *Every coordinate system* x^ν *for which*
a) *the point* O *has the coordinates* $x^\nu = 0$,
b) *the equations* (14.3) *hold*
will be referred to as an admissible coordinate system.

NOTE. If x^ν and $x^{\nu'}$ are two admissible coordinate systems, we assume that the transformation $x^{\nu'} \leftrightarrow x^\nu$ may be expanded in a Taylor series

$$(14.4)\text{a} \qquad x^{\nu'} = C_\lambda{}^{\nu'} x^\lambda + \tfrac{1}{2} C_\lambda{}^{\nu'}{}_\mu x^\lambda x^\mu + \ldots,$$

$$(14.4)\text{b} \qquad x^\nu = C_{\lambda'}{}^\nu x^{\lambda'} + \tfrac{1}{2} C_{\lambda'}{}^\nu{}_{\mu'} x^{\lambda'} x^{\mu'} + \ldots.$$

In this case the C's are constants and symmetric in their lower indices, and in particular,

$$(14.4)\text{c} \qquad \begin{gathered} {}^0h_{\lambda\mu} = C_\lambda{}^{\alpha'} C_\mu{}^{\beta'} \delta^{\omega\nu}_{\alpha'\beta'}\, {}^0h_{\omega\nu}{}^{1} \\ {}^0h_{\alpha\beta} \delta^{\alpha\,\beta}_{\omega'\nu'} = C_{\omega'}{}^\alpha C_{\nu'}{}^\beta\, {}^0h_{\alpha\beta} \\ \delta^{\nu'}_4 = C_4{}^{\nu'}, \quad \delta^\nu_{4'} = C_{4'}{}^\nu \\ C_{\lambda'}{}^\nu{}_{\mu'} = -\, C_{\lambda'}{}^\alpha C_{\mu'}{}^\beta C_{\gamma'}{}^\nu C_\alpha{}^{\gamma'}{}_\beta. \end{gathered}$$

LEMMA (14.2). *If* x^ν *and* $x^{\nu'}$ *are two admissible coordinate systems, then we have*

$$(14.5)\text{a} \qquad \partial_{\omega'} k_{\mu'\lambda'} \overset{0}{=} C_{\omega'}{}^\alpha C_{\mu'}{}^\beta C_{\lambda'}{}^\gamma \partial_\alpha k_{\beta\gamma}$$

$$(14.5)\text{b} \qquad (\partial_{\xi'} k^{\beta'\alpha'})\,(\partial_{\beta'} k_{\alpha'}{}^{\nu'}) \overset{0}{=} C_{\xi'}{}^\gamma C_\delta{}^{\nu'} (\partial_\gamma k^{\beta\alpha})\,(\partial_\beta k_\alpha{}^\delta).$$

PROOF. One obtains first

$$\partial_{\omega'} k_{\mu'\lambda'} = \frac{\partial x^\alpha}{\partial x^{\omega'}} \left[\left(\partial_\alpha \frac{\partial x^\beta}{\partial x^{\mu'}} \, \frac{\partial x^\gamma}{\partial x^{\lambda'}} \right) k_{\beta\gamma} + \frac{\partial x^\beta}{\partial x^{\mu'}} \, \frac{\partial x^\gamma}{\partial x^{\lambda'}} \partial_\alpha k_{\beta\gamma} \right],$$

---

[1] Here we put

$$\delta^{\nu'}_\lambda = \delta^\nu_{\lambda'} = \delta^\nu_\lambda, \qquad \delta^{\omega\,\nu}_{\alpha'\beta'} = \delta^\omega_{\alpha'} \delta^\nu_{\beta'}.$$

for any coordinate systems x^ν and $x^{\nu'}$, so that by virtue of (14.1)b and (14.4)b, we obtain (14.5)a. The equation (14.5)b may be proved in the same way.

In the next Lemma we use the following abbreviations:

$$(14.6)\text{a} \qquad \mathscr{K}_{\omega\mu\nu} \overset{\text{def}}{=} (\partial_\omega k_{\nu\mu} + \partial_\mu k_{\omega\nu} + \partial_\nu k_{\omega\mu})_O,$$

$$(14.6)\text{b} \qquad \mathscr{U}_{\nu\omega\mu\xi} \overset{\text{def}}{=} -(\partial_\xi k_{(\mu}{}^\alpha)_O \mathscr{K}_{\omega)\nu\alpha}.$$

LEMMA (14.3). *The tensor $U_{\nu\omega\mu}$ defined by* II (18.3) *satisfies the following conditions*

$$(14.7)\text{a} \qquad U_{\nu\omega\mu} \overset{0}{=} 0,$$

$$(14.7)\text{b} \qquad U_{\nu\omega\mu} \overset{1}{=} x^\xi \mathscr{U}_{\nu\omega\mu\xi}.$$

Moreover, if x^ν and $x^{\nu'}$ are two admissible coordinate systems, then

$$(14.8) \qquad \mathscr{U}_{\nu'\omega'\mu'\xi'} = C_{\nu'}{}^\alpha C_{\omega'}{}^\beta C_{\mu'}{}^\gamma C_{\xi'}{}^\delta \mathscr{U}_{\alpha\beta\gamma\delta}.$$

PROOF. The equations (14.7) follow immediately from II (18.3) and (14.1)a,b. The equation (14.8) is an immediate consequence of (14.5)a.

The last of the lemmas we shall need in the next section is

LEMMA (14.4). *We have*

$$(14.9)\text{a} \qquad v^\lambda D_\lambda v^\nu \overset{0}{=} v^\lambda \nabla_\lambda v^\nu,$$

$$(14.9)\text{b} \qquad v^\lambda D_\lambda v^\nu \overset{1}{=} (v^\lambda \nabla_\lambda v^\nu)_O + x^\xi[(\partial_\xi v^\lambda \nabla_\lambda v^\nu)_O + \mathscr{U}^\nu{}_{\omega\mu\xi}(v^\omega v^\mu)_O].$$

PROOF. We have

$$(14.10) \qquad v^\lambda D_\lambda v^\nu \overset{1}{=} (v^\lambda D_\lambda v^\nu)_O + x^\xi[\partial_\xi(v^\lambda D_\lambda v^\nu)]_O,$$

and according to Lemma (14.3)

$$(14.11)\text{a} \qquad (v^\lambda D_\lambda v^\nu)_O = (v^\lambda \nabla_\lambda v^\nu)_O$$

$$(14.11)\text{b} \qquad [\partial_\xi(v^\lambda D_\lambda v^\nu)]_O = [\partial_\xi(v^\lambda \nabla_\lambda v^\nu)]_O + \mathscr{U}^\nu{}_{\lambda\mu\xi}(v^\lambda v^\mu)_O.$$

The equations (14.9) follow from (14.10) and (14.11).

15. Newtonian laws.

THEOREM (15.1). *There is always an admissible coordinate system for which*

$$(15.1)\text{a} \qquad \partial_4 h_{\lambda 4} \overset{0}{=} 0,$$

$$(15.1)\text{b} \qquad \partial_\xi(v^\lambda \nabla_\lambda v^\nu) + \mathscr{U}^\nu{}_{\omega\mu\xi} v^\omega v^\mu + (\partial_\xi k^{\beta\alpha})(\partial_\beta k_\alpha{}^\nu) \overset{0}{=} 0.$$

PROOF. Let $x^{\nu'}$, x^{ν} be two admissible coordinate systems. Then a simple inspection based on (14.1), (14.3) and (14.4) leads to

$$(15.2)\qquad \partial_{4'}h_{\lambda'4'} \overset{0}{=} -C_{4'}{}^{4}{}_{\lambda'} + C_{\lambda'}{}^{\alpha}C_{4'}{}^{\beta}{}_{4'}\,{}^{0}h_{\alpha\beta} + C_{\lambda'}{}^{\alpha}(\partial_4 h_{\alpha 4})_O,$$

and in particular,

$$(15.3)\text{a}\qquad \partial_{4'}h_{4'4'} \overset{0}{=} -2C_{4'}{}^{4}{}_{4'} + (\partial_4 h_{44})_O$$

$$(15.3)\text{b}\qquad \partial_{4'}h_{i'4'} \overset{0}{=} C_{i'}{}^{a}\,[C_{4'}{}^{b}{}_{4'}\,{}^{0}h_{ab} + (\partial_4 h_{a4})_O] - C_{4'}{}^{4}{}_{i'}.$$

Hence, if we put

$$(15.4)\text{a}\qquad 2C_{4'}{}^{4}{}_{4'} \overset{\text{def}}{=} (\partial_4 h_{44})_O$$

$$(15.4)\text{b}\qquad C_{4'}{}^{4}{}_{i'} \overset{\text{def}}{=} C_{i'}{}^{a}[C_{4'}{}^{b}{}_{4'}\,{}^{0}h_{ab} + (\partial_4 h_{a4})_O],$$

we see from (15.3) that the left-hand term of (15.2) is equal to zero at O. Omitting the primes we obtain (15.1)a. In order to get (15.1)b let us introduce the abbreviations

$$P^{\nu} \overset{\text{def}}{=} v^{\lambda}\nabla_{\lambda}v^{\nu}$$

$$\mathscr{X}_{\xi}{}^{\nu} \overset{\text{def}}{=} \mathscr{U}^{\nu}{}_{\omega\mu\xi}v^{\omega}v^{\mu} + (\partial_{\xi}k^{\beta\alpha})\,(\partial_{\beta}k_{\alpha}{}^{\nu}).$$

The left-hand term in (15.1)b is the value of

$$\partial_{\xi}P^{\nu} + \mathscr{X}_{\xi}{}^{\nu}.$$

Therefore, we obtain by virtue of Lemmas (14.2) and (14.3)

$$(15.5)\qquad \partial_{\xi'}P^{\nu'} + \mathscr{X}_{\xi'}{}^{\nu'} \overset{0}{=} C_{\xi'}{}^{\alpha}\{P^{\beta}C_{\alpha}{}^{\nu'}{}_{\beta} + C_{\beta}{}^{\nu'}(\partial_{\alpha}P^{\beta} + \mathscr{X}_{\alpha}{}^{\beta})\}_O.$$

Let us consider the conditions

$$(P^{\beta})_O C_{\alpha}{}^{\nu'}{}_{\beta} = -C_{\beta}{}^{\nu'}(\partial_{\alpha}P^{\beta} + \mathscr{X}_{\alpha}{}^{\beta})_O,$$

equivalent to

$$(15.6)\qquad (P^{\beta})_0 C_{\beta}{}^{\beta'}C_{\mu'}{}^{\nu}{}_{\beta'} = C_{\mu'}{}^{\alpha}(\partial_{\alpha}P^{\nu} + \mathscr{X}_{\alpha}{}^{\nu})_O,$$ [1]

for which the left-hand term in (15.5) vanishes at O. The equation (15.6) yields for $\nu = 4$, $\mu' = 4'$

$$(15.7)\text{a}\qquad (P^4)_O C_{4'}{}^{4}{}_{4'} + (P^b)_O C_b{}^{b'}C_{4'}{}^{4}{}_{b'} = (\partial_4 P^4 + \mathscr{X}_4{}^4)_O,$$

or according to (15.4)

$$(15.7)\text{b}\qquad \tfrac{1}{2}(P^4\partial_4 h_{44})_O + (P^a)_O[C_{4'}{}^{b}{}_{4'}\,{}^{0}h_{ab} + (\partial_4 h_{a4})_O] = (\partial_4 P^4 + \mathscr{X}_4{}^4)_O.$$

---

[1] *Cf.* the last of equations (14.4)c.

This is a condition for the coefficient $C_{4'}{}^{b}{}_{4'}$, which was not defined by (15.4). Hence, if we choose $C_{4'}{}^{b}{}_{4'}$ according to (15.7)b, then (15.7)a is satisfied for $C_{b'}{}^{4}{}_{4'}$ defined by (15.4). The remaining $C_{\mu'}{}^{\nu}{}_{\lambda'}$ may then be obtained from (15.6) (which does not define them in a unique way). Hence, with this choice the left-hand term in (15.5) vanishes at O. Omitting the primes, we obtain (15.1)b.

DEFINITION (15.1). *Every admissible coordinate system for which* (15.1) *holds will be referred to as an allowable system for the third class.*

In order to formulate the main Theorem of this section we remind the reader that the electromagnetic field may be introduced even in the gravitational theory based on $h_{\lambda\mu}$. Then the electromagnetic force vector F^{ν} is defined by

$$F^{\nu} \overset{\text{def}}{=} \nabla_{\lambda} M^{\lambda\nu}, \tag{15.8}$$

where the symmetric tensor $M^{\lambda\nu}$ is the corresponding momentum energy tensor of electromagnetism

$$M_{\lambda}{}^{\nu} \overset{\text{def}}{=} f^{\nu\alpha} f_{\alpha\lambda} + \tfrac{1}{4}\delta^{\nu}_{\lambda} f^{\alpha\beta} f_{\alpha\beta}. \tag{15.9}$$

We shall call F^{ν} the *classical relativistic electromagnetic force vector.* By the words *classical relativistic,* we stress the fact that we are dealing with the gravitational theory based on $h_{\lambda\mu}$. Moreover, the equality sign $\overset{0}{=}$ $(\overset{1}{=})$ will be described as "equal at O" ("equal in the linear neighborhood of O").

THEOREM (15.2). *Every allowable coordinate system for the third class x^{ν} has the following properties:*

a) *The unified law of inertia* (13.1) *expressed in x^{ν} reduces at O to the classical Newtonian law of gravitation.*
b) *The unified law of inertia* (13.1) *expressed in x^{ν} reduces in the linear neighborhood of O to the classical relativistic second law of Newton for the classical relativistic electromagnetic force vector F^{ν} defined by* (15.8).

PROOF. Let us first prove the assertion (a). We obtain from (13.1) and (14.9)a

$$(v^{\lambda}\nabla_{\lambda} v^{\nu})_{O} = \left(\frac{d^2x^{\nu}}{dq^2} + \{\lambda{}^{\nu}{}_{\mu}\}\frac{dx^{\lambda}}{dq}\frac{dx^{\mu}}{dq}\right)_{O} = 0. \tag{15.10}$$

Let us now identify t with x^4 so that we have according to (14.3)a

$$(15.11) \qquad \frac{dx^\nu}{dt} \overset{0}{=} \delta_4^\nu.$$

Moreover, we have from (14.2)

$$\left(\frac{dt}{dq}\right)^2 = -\gamma^2/h_{\lambda\mu}\frac{dx^\lambda}{dt}\frac{dx^\mu}{dt},$$

and therefore, according to (15.11) and (15.1)a

$$(15.12) \qquad \frac{d^2t}{dq^2} \overset{0}{=} 0.$$

Therefore, the equations (15.10) may be rewritten as follows

$$(15.13)\text{a} \qquad \frac{d^2x^a}{dt^2} + \{{}_4{}^a{}_4\} \overset{0}{=} 0,$$

$$(15.13)\text{b} \qquad \{{}_4{}^4{}_4\} \overset{0}{=} 0.$$

Now we have according to (14.3)b and (15.1)a

$$(15.14)\text{a} \quad \{{}_4{}^a{}_4\} \overset{0}{=} \tfrac{1}{2}{}^0h^{ab}(2\partial_4 h_{b4} - \partial_b h_{44})_O = -\tfrac{1}{2}{}^0h^{ab}(\partial_b h_{44})_O,$$

$$(15.14)\text{b} \qquad \{{}_4{}^4{}_4\} \overset{0}{=} \tfrac{1}{2}{}^0h^{44}(\partial_4 h_{44})_O = 0.$$

The equation (15.13)b reduces to an identity by virtue of (15.14)b. On the other hand, if we put

$$\Phi \overset{\text{def}}{=} -\tfrac{1}{2}h_{44},$$

we obtain from (15.13)a and (15.14)a

$$(15.15) \qquad \frac{d^2x^a}{dt^2} + h^{ab}\partial_b\Phi \overset{0}{=} 0.$$

If we replace the symbol $\overset{0}{=}$ by the symbol "$=$" we obtain the classical Newtonian law of gravitation (the force vector $-\partial_b\Phi$ acting on a particle of unit mass). This proves the assertion (a).

In order to prove assertion (b), we first observe that by virtue of (1.3)a,b we obtain from (15.9)

$$M_\lambda{}^\nu = k_{\alpha\lambda}k^{\nu\alpha},$$

so that

$$(15.16)\text{a} \qquad F^\nu = \nabla_\lambda\, k_\alpha{}^\lambda k^{\nu\alpha}.$$

This equation can be rewritten. In fact, the equation (3.1)a is equivalent by virtue of (1.3)a to

$$\nabla_\lambda k_\alpha{}^\lambda = 0. \tag{15.17}$$

Hence, (15.16)a can be written

$$F^\nu = k^{\beta\alpha} \nabla_\beta k_\alpha{}^\nu, \tag{15.16)b}$$

and yields, because of (14.1)b,

$$F^\nu \overset{1}{=} x^\xi (\partial_\xi k^{\beta\alpha})_O \, (\partial_\beta k_\alpha{}^\nu)_O. \tag{15.18}$$

On the other hand, we obtain from (13.1) and (14.9)b

$$(v^\lambda \nabla_\lambda v^\nu)_O + x^\xi [(\partial_\xi v^\lambda \nabla_\lambda v^\nu) + \mathcal{U}^\nu{}_{\omega\mu\xi} v^\omega v^\mu]_O \overset{1}{=} 0. \tag{15.19}$$

If we denote by s the arc length of our trajectory, then we have from (14.2)

$$\left(\frac{ds}{dq}\right)^2 = \gamma^2,$$

so that

$$\frac{dx^\nu}{dq} = \frac{dx^\nu}{ds}\gamma, \quad \frac{d^2x^\nu}{dq^2} = \frac{d^2x^\nu}{ds^2}\gamma^2.$$

Therefore,

$$v^\lambda \nabla_\lambda v^\nu = \gamma^2 \frac{dx^\lambda}{ds} \nabla_\lambda \frac{dx^\nu}{ds}. \tag{15.20}$$

On the other hand, we have from (15.18) and (15.1)b

$$x^\xi [(\partial_\xi v^\lambda \nabla_\lambda v^\nu) + \mathcal{U}^\nu{}_{\omega\mu\xi} v^\omega v^\mu]_O \overset{1}{=} -F^\nu \tag{15.21}$$

so that (15.19) can be rewritten

$$\gamma^2 \frac{dx^\lambda}{ds} \nabla_\lambda \frac{dx^\nu}{ds} \overset{1}{=} F^\nu. \tag{15.22}$$

If we replace the symbol "$\overset{1}{=}$" in (15.22) by the symbol "$=$", we obtain the classical relativistic second Newton law for the electromagnetic force vector F^ν. This proves the assertion (b) [1].

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[1] A similar result has been obtained by Bonnor, W. B., (5), who also started by changing the original set of Einstein's conditions. For another approach see Gallaway, J., (1); Moffat, J., (2); Narlikar, V.V. — Rao, B.R., (1); Infeld, L., (1), (3).

NOTE. Theorem (15.2) requires the existence of a point O for which (14.1)b holds.[1] A similar theorem can be proved without involving such a preferred point. Then one has to impose some differential conditions on $g_{\lambda\mu}$ at an arbitrary point. This point of view has been adopted in Hlavatý, V., (17).

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[1] Of course we must have $k_{\lambda\mu} \neq 0$ in the neighborhood of O.

CHAPTER V

PART II

First two classes

16. Prerequisites. [1] In this second part we shall consider the physical interpretation of the field equations

$$S_\lambda = 0 \tag{16.1a}$$

$$\partial_{[\omega} X_{\mu]} = R_{[\omega\mu]} \tag{16.1b}$$

$$R_{(\mu\lambda)} = 0 \tag{16.1c}$$

for the first two classes. According to Definition I (1.1) the first class is characterized by

$$k \neq 0, \tag{16.2a}$$

while the second class is characterized by

$$\text{(16.2)} \qquad \text{b)} \quad K \neq 0, \qquad \text{c)} \quad k = 0.$$

All subsequent results will be formulated for the most general case, so that the results for particular cases characterized by (16.2)b,c may be obtained by simple substitution of these conditions.

The relationship between $f_{\lambda\mu}$ and $k_{\lambda\mu}$ is given by I (15.2), *i.e.*,

$$f_{\lambda\nu} = \frac{\sigma}{\sqrt{g}}\left(\kappa\sqrt{|k|}\,k_{\lambda\nu} - \tfrac{1}{2}e_{\lambda\nu\alpha\beta}k^{\alpha\beta}\right) \tag{16.3a}$$

$$f^{\lambda\nu} = \frac{\sigma}{\sqrt{g}}\left(\kappa\sqrt{|k|}\,k^{\lambda\nu} + \tfrac{1}{2}E^{\lambda\nu\alpha\beta}k_{\alpha\beta}\right). \tag{16.3b}$$

According to Agreement I (15.1) and Theorem I (15.5), the equations (16.3) admit a unique solution $k_{\lambda\nu}$ given by I (15.17). Introducing the abbreviations

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[1] In this part we follow the general trend of the first part where all problems were dealt with in detail. Therefore, we shall keep this part somewhat sketchy, referring very frequently to corresponding problems in the first part.

(16.4)a $$G \overset{\text{def}}{=} 1/e = 1/(1 + \tfrac{1}{2} f_{\alpha\beta} f^{\alpha\beta} + f)$$

(16.4)b $$F^{\lambda\nu} \overset{\text{def}}{=} \sqrt{G} f^{\lambda\nu}$$

(16.4)c $$L_{\lambda\nu} \overset{\text{def}}{=} - \varphi\sqrt{|f|}\, F_{\lambda\nu},$$ [1]

we may write the solution of (16.3)a,b in the form

(16.3)c $$\sigma k_{\lambda\nu} = \tfrac{1}{2} e_{\lambda\nu\alpha\beta} F^{\alpha\beta} + L_{\lambda\nu}$$

(16.3)d $$\sigma k^{\lambda\nu} = - \tfrac{1}{2} E^{\lambda\nu\alpha\beta} F_{\alpha\beta} + L^{\lambda\nu}.$$

The following abbreviation will be very useful. If A_ω, $B_{\mu\lambda}$ are two objects, we put

(16.5)a $$A_{\{\omega} B_{\mu\lambda\}} \overset{\text{def}}{=} A_\omega B_{\lambda\mu} + A_\mu B_{\omega\lambda} + A_\lambda B_{\omega\mu}.$$

Thus, if $B_{\mu\lambda}$ is skew symmetric,

(16.5)b $$A_{\{\omega} B_{\mu\lambda\}} = - 3 A_{[\omega} B_{\mu\lambda]} + 2 A_\lambda B_{\omega\mu}.$$

DEFINITION (16.1). *We put*

(16.6)a $$W^\nu \sqrt{|\mathfrak{h}|} \overset{\text{def}}{=} \partial_\lambda F^{\lambda\nu} \sqrt{|\mathfrak{h}|} \quad (= \sqrt{|\mathfrak{h}|}\, \nabla_\lambda F^{\lambda\nu})$$

(16.6)b $$w^\nu \sqrt{|\mathfrak{h}|} \overset{\text{def}}{=} \partial_\lambda f^{\lambda\nu} \sqrt{|\mathfrak{h}|} \quad (= \sqrt{|\mathfrak{h}|}\, \nabla_\lambda f^{\lambda\nu})$$

(16.7) $$G_\lambda \overset{\text{def}}{=} \tfrac{1}{2} \partial_\lambda \ln G \quad (= U_\lambda - \partial_\lambda \ln (1 - f)).$$ [2]

THEOREM (16.1). *We have*

(16.8) $$W^\nu = \sqrt{G}(w^\nu + G_\lambda f^{\lambda\nu}) = \sqrt{G} w^\nu + G_\lambda F^{\lambda\nu},$$

(16.9)a $$\partial_\nu \sqrt{|\mathfrak{h}|}\, W^\nu = 0,$$

(16.9)b $$\partial_\nu \sqrt{|\mathfrak{h}|}\, w^\nu = 0.$$

PROOF. (16.8) follows easily from (16.4)b and Definition (16.1). (16.9) follows from (16.6).

We know (*Cf.* Theorem IV (4.1)) that (16.1)a is equivalent to

(16.10)a $$\partial_{[\omega} f_{\mu\lambda]} = 0.$$

In the case of the first class $f_{\lambda\mu}$ is the only Maxwell field [3] belonging to $g_{\lambda\mu}$ (*Cf.* Theorem IV (4.2)). The equations (16.10)a are equivalent to

---

[1] See also I (15.3), I (15.4), I (15.7), I (15.19) and (12.3). Note that for the second class $G = g$.

[2] *Cf.* I (15.19) and II (7.1)c.

[3] up to a multiplicative constant.

$$f_{\mu\lambda} = \partial_{[\mu} f_{\lambda]}. \tag{16.10)b}$$

Thus, six unknowns $k_{\lambda\mu}$ are expressible in terms of four unknowns f_λ.[1]

17. Tensors $K_{\omega\mu\lambda}$, $S_{\omega\mu\lambda}$, and $U_{\nu\lambda\mu}$.

THEOREM (17.1). *The tensor* $K_{\omega\mu\lambda}$ *defined by* II (3.5) *may be expressed in terms of* $f_{\lambda\mu}$ *and* w^ν *as follows:*

$$\sigma K_{\omega\mu\lambda} = \sqrt{G} w^\nu e_{\nu\omega\mu\lambda} + {}^0K_{\omega\mu\lambda}. \tag{17.1)a}$$

The tensor

$$\sigma\, {}^0K_{\omega\mu\lambda} \overset{\text{def}}{=} \sigma e_{\omega\mu\alpha\beta}(\nabla_\lambda F^{\alpha\beta} + \delta^\alpha_\lambda F^{\beta\gamma} G_\gamma) + \sigma \nabla_{\{\omega} L_{\mu\lambda\}} \tag{17.1)b}$$

does not contain w^ν.

PROOF. We have, according to II (3.5), (16.5)a,b and (16.3)c,

$$\sigma K_{\omega\mu\lambda} = \sigma \nabla_{\{\omega} k_{\mu\lambda\}} = \tfrac{1}{2} \nabla_{\{\omega} e_{\mu\lambda\}\alpha\beta} F^{\alpha\beta} + \nabla_{\{\omega} L_{\mu\lambda\}}. \tag{17.2}$$

On the other hand, by (16.5)b

$$\tfrac{1}{2} \nabla_{\{\omega} e_{\mu\lambda\}\alpha\beta} F^{\alpha\beta} = \tfrac{3}{2}\, e_{\alpha\beta[\lambda\mu} \nabla_{\omega]} F^{\alpha\beta} + e_{\omega\mu\alpha\beta} \nabla_\lambda F^{\alpha\beta}. \tag{17.3)a}$$

If we put

$$X_{\lambda\mu\omega} \overset{\text{def}}{=} \tfrac{3}{2}\, e_{\alpha\beta[\lambda\mu} \nabla_{\omega]} F^{\alpha\beta},$$

then, according to I (2.12) c,d and (16.6)a,

$$\begin{aligned} E^{\lambda\mu\omega\nu} X_{\lambda\mu\omega} &= 6W^\nu \\ e_{\lambda\mu\omega\nu} E^{\xi\eta\zeta\nu} X_{\xi\eta\zeta} &= 6X_{\lambda\mu\omega} = 6W^\nu e_{\lambda\mu\omega\nu} \\ &= 6W^\nu e_{\nu\omega\mu\lambda}. \end{aligned}$$

Thus, (17.3)a reduces by virtue of (16.8) to

$$\begin{aligned} \tfrac{1}{2} \nabla_{\{\omega} e_{\mu\lambda\}\alpha\beta} F^{\alpha\beta} = \sqrt{G} w^\nu e_{\nu\omega\mu\lambda} &+ G_\alpha F^{\alpha\nu} e_{\nu\mu\omega\lambda} \\ &+ e_{\omega\mu\alpha\beta} \nabla_\lambda F^{\alpha\beta}. \end{aligned} \tag{17.3)b}$$

The equation (17.1)a results from (17.2) and (17.3)b. The equation (17.1)b shows that $\sigma\, {}^0K_{\omega\mu\lambda}$ does not contain w^ν.

---

[1] If one admits restrictions other than (16.1), one obtains another identification. *Cf.* Hély, J., (3).

In the next Theorem we use the tensor $Z^{\alpha\beta\gamma}_{\omega\mu\nu}$ defined by II (10.3), and denote by $S^{\alpha\beta\gamma}_{\omega\mu\nu}$ the tensor

$$S^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\text{def}}{=} Z^{\alpha\beta\gamma}_{\omega\mu\nu} + 3Z^{\alpha\beta\gamma}_{[\xi\eta\nu]}k_{\omega}{}^{\xi}k_{\mu}{}^{\eta}, \tag{17.4}$$

where we assume that the right-hand term is expressed in terms of $f_{\lambda\nu}$ by means of (16.3)c,d. It is obvious that $S^{\alpha\beta\gamma}_{\omega\mu\nu}$ does not contain w^{ν}.

THEOREM (17.2). *Put*[1]

$$2\,{}^{0}S_{\omega\mu\nu} = {}^{0}K_{\alpha\beta\gamma}S^{\alpha\beta\gamma}_{\omega\mu\nu}. \tag{17.5)a}$$

Then

$$2\sigma S_{\omega\mu\nu} = \sqrt{G}w^{\delta}e_{\delta\alpha\beta\gamma}S^{\alpha\beta\gamma}_{\omega\mu\nu} + 2\,{}^{0}S_{\omega\mu\nu}. \tag{17.5)b}$$

The tensor $\sigma\,{}^{0}S_{\omega\mu\nu}$ *does not contain* w^{ν}.

PROOF. One finds from II (10.6)c and II (11.1)

$$\begin{aligned} 2S_{\omega\mu\nu} &= K_{\alpha\beta\gamma}(Z^{\alpha\beta\gamma}_{\omega\mu\nu} + 3Z^{\alpha\beta\gamma}_{[\xi\eta\nu]}k_{\omega}{}^{\xi}k_{\mu}{}^{\eta}) \\ &= K_{\alpha\beta\gamma}S^{\alpha\beta\gamma}_{\omega\mu\nu}. \end{aligned}$$

The remaining steps of the proof based on (17.1) are almost obvious.

In the next Theorem we denote by $U^{\alpha\beta\gamma}_{\nu\lambda\mu}$ the tensor defined by

$$U^{\alpha\beta\gamma}_{\nu\lambda\mu} \overset{\text{def}}{=} -\,k_{(\lambda}{}^{\omega}Z^{\alpha\beta\gamma}_{\mu)\nu\omega}, \tag{17.6}$$

where the right hand term is assumed to be expressed by means of $f_{\lambda\mu}$.

THEOREM (17.3). *Put*

$${}^{0}U_{\nu\lambda\mu} \overset{\text{def}}{=} U^{\alpha\beta\gamma}_{\nu\lambda\mu}\,{}^{0}K_{\alpha\beta\gamma}. \tag{17.7)a}$$

Then

$$\sigma U_{\nu\lambda\mu} = \sqrt{G}w^{\delta}e_{\delta\alpha\beta\gamma}U^{\alpha\beta\gamma}_{\nu\lambda\mu} + {}^{0}U_{\nu\lambda\mu}. \tag{17.7)b}$$

The tensor $\sigma\,{}^{0}U_{\nu\lambda\mu}$ *does not contain* w^{ν}.

The proof follows easily from II (10.6)c, II (11.7), (17.6) and (17.1).

NOTE. Compute U_{μ} from (17.7)b and substitute it into (16.7).

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[1] We deliberately use in (17.5) the same symbols as in (2.2), although they have a different meaning here. A similar remark also holds for (17.7) and (2.3).

The result is

$$(17.8)\text{a}\qquad \sigma(\sqrt{G}w^{\delta}e_{\delta\alpha\beta\gamma}U^{\alpha\beta\gamma}_{\nu\lambda\mu}h^{\nu\lambda}+{}^{0}U^{\lambda}{}_{\lambda\mu})-G_{\mu}=\partial_{\mu}\ln(1-f).$$

This equation does not restrict w^{ν} as defined by (16.6)b. Rather, this latter equation converts (17.8)a into an identity. In any case (17.8)a is equivalent to

$$(17.8)\text{b}\qquad \sigma(\sqrt{G}w^{\delta}e_{\delta\alpha\beta\gamma}U^{\alpha\beta\gamma}_{\nu\lambda\mu}h^{\nu\lambda}+{}^{0}U^{\lambda}{}_{\lambda\mu})=U_{\mu}$$

as follows from (16.7) [1]. It is irrelevant for our purposes whether we express U_{μ} by means of (17.8)b or by II (7.1)c. We shall always use the latter equation.

THEOREM (17.4). *The connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ given by* II (3.1) *may be expressed as follows:*

$$(17.9)\qquad \Gamma_{\lambda}{}^{\nu}{}_{\mu}=\{\lambda{}^{\nu}{}_{\mu}\}+\frac{\sigma}{2}\sqrt{G}w^{\delta}e_{\delta\alpha\beta\gamma}(S^{\alpha\beta\gamma}_{\lambda\mu\xi}+2U^{\alpha\beta\gamma}_{\xi\lambda\mu})h^{\xi\nu}$$
$$+\sigma({}^{0}S_{\lambda\mu}{}^{\nu}+{}^{0}U^{\nu}{}_{\lambda\mu}).$$

The proof follows easily from II (3.1), (17.5)b and (17.7)b.

18. Maxwell's equations.

In the next Theorem we shall use the tensor $S_{\omega\mu}$ defined by (4.2).

THEOREM (18.1). *Neither one of the conditions* (16.1)b,c, (16.2)c *yields*

$$(18.1)\qquad S_{\omega\mu}=0.$$

The proof follows the pattern of the corresponding part of the proof of Theorem (4.1).

THEOREM (18.2). *The tensor $^{*}A_{[\omega\mu]}$ defined by* IV (3.12)a *may be expressed in terms of $f_{\lambda\mu}$ and w^{ν} as follows:*

$$(18.2)\qquad {}^{*}A_{[\omega\mu]}={}^{0}A_{\omega\mu}+A_{\omega\mu\lambda}\sqrt{G}w^{\lambda}+A_{\omega\mu\lambda\nu}Gw^{\lambda}w^{\nu}$$
$$+\tfrac{1}{2}a_{\omega\mu\lambda\nu}\nabla^{\lambda}\sqrt{G}w^{\nu}$$

where

$$(18.3)\qquad \tfrac{1}{2}a_{\omega\mu\lambda\nu}\overset{\text{def}}{=}2e_{\nu\alpha\beta\gamma}U^{\alpha\beta\gamma}_{\delta\lambda[\omega}(\tfrac{1}{2}e_{\mu]\xi\eta\zeta}F^{\eta\zeta}+L_{\mu]\xi})h^{\xi\delta}$$

and the tensors $^{0}A_{\omega\mu}$, $A_{\omega\mu\lambda}$, $A_{\omega\mu\lambda\nu}$ have properties similar to those mentioned in Theorem (4.2).

---

[1] (17.8)b can also be obtained by contracting (17.7)b.

The proof is similar to the proof of Theorem (4.2).

THEOREM (18.3). *The system* (16.1)b *of field equations is equivalent to*

$$\partial_{[\omega} X_{\mu]} = \frac{\sigma}{2} e_{\nu\omega\mu\lambda} \nabla^{\lambda} \sqrt{G} w^{\nu} + \frac{\sigma}{2} \nabla^{\alpha}\,{}^{0}K_{\omega\mu\alpha} + {}^{*}A_{[\omega\mu]} + S_{\omega\mu}. \tag{18.4)a}$$

The systems (16.1)a,b *of field equations are equivalent to* (16.10)a *and*

$$\partial_{[\omega} X_{\mu]} = \frac{\sigma}{2} e_{\nu\omega\mu\lambda} \nabla^{\lambda} \sqrt{G} w^{\nu} + \frac{\sigma}{2} \nabla^{\alpha}\,{}^{0}K_{\omega\mu\alpha} + {}^{*}A_{[\omega\mu]}. \tag{18.4)b}$$

In both cases ${}^{*}A_{[\omega\mu]}$ *is to be taken from* (18.2), *and* $S_{\omega\mu}$ *is defined by* (4.2).

The proof based on IV (3.13)a, (17.1) and Theorem (18.1) follows the pattern of the proofs of Theorems (4.3)a,b.

The Note following Theorem (4.3)b holds here also. In particular, we may use (4.8) in a slightly different form

$$f_{\lambda\mu} = f_{\lambda\mu}(X_{\cdot}, h_{\cdot\cdot}, w^{\cdot}, \nabla_{\cdot} w^{\cdot}). \tag{18.5}$$

THEOREM (18.4). *Upon substituting from* (18.5) *into the equations* (16.6)b *which define* w^{ν}, *one obtains four partial differential equations*

$$w^{\nu}\sqrt{|\mathfrak{h}|} = \partial_{\lambda} f^{\lambda\nu}\sqrt{|\mathfrak{h}|} \tag{18.6}$$

of the second order for four unknowns w^{ν}.

The proof is similar to the proof of Theorem (4.4).

NOTE I. Substituting into (18.5) the solution of (18.6),

$$w^{\nu} = w^{\nu}(X_{\cdot}, h_{\cdot\cdot}), \tag{18.7}$$

we obtain $f_{\lambda\mu}$ expressed in terms of X_{λ} and $h_{\lambda\mu}$

$$f_{\lambda\mu} = f_{\lambda\mu}(X_{\cdot}, h_{\cdot\cdot}). \tag{18.8}$$

NOTE II. A procedure similar to the one exhibited in Note II following Theorem (4.4) leads here to (18.6) also. In order to give to (16.6)b an operational meaning *we use* (18.4)b *as another definition for* w^{ν}. Substituting w^{ν}, defined by (18.4)b, into (16.6)b we convert (16.6)b into the set (18.6) of four partial differential equations for $f_{\lambda\mu}$: the left hand term is a function of $f_{\lambda\mu}$ defined by (18.4)b.

NOTE III. From the manner in which we obtained the equations

$$\partial_{[\omega}f_{\mu\lambda]} = 0 \tag*{(18.9)a}$$

$$\mathfrak{w}^{\nu}\sqrt{|\mathfrak{h}|} = \partial_{\lambda}f^{\lambda\nu}\sqrt{|\mathfrak{h}|}, \tag*{(18.9)b}$$

it is clear that *they result as an immediate consequence of the field equations* (16.1)a,b. The form of (18.9) justifies the following

AGREEMENT (18.1). *We shall confine ourselves to the case*

$$\mathfrak{w}^2 \overset{\text{def}}{=} \mathfrak{w}^{\nu}\mathfrak{w}_{\nu}|\mathfrak{h}| \leq 0 \tag*{(18.10)}$$

and identify

a) $f_{\lambda\mu}$ *with the electromagnetic field,*
b) $\mathfrak{w}^{\nu}\sqrt{|\mathfrak{h}|}$ *with the current density vector,*
c) (18.9) *with Maxwell's equations,*
d) (16.9)b [1] *with the continuity equation of the charge and current,*
e) $\mathfrak{w} = \pm\sqrt{|\mathfrak{w}^2|}$ *with the electric charge density whever* $\mathfrak{w}^2 \neq 0$. [2]

NOTE. The tensors ${}^{*}k^{\lambda\nu}\sqrt{-\mathfrak{g}}$ and $k_{\lambda\nu}$ may be identified with the (B, E)-tensor and (H, D)-tensor respectively in the Born *non-linear* electrodynamics. [3] Then the current vector density has to be defined by means of $\mathfrak{E}^{\omega\mu\lambda\nu}S_{\omega\mu\lambda}$.

19. Generalized Maxwell's set.

THEOREM (19.1). *A necessary and sufficient condition for pure radiation*

$$\mathfrak{w}^{\nu} = 0 \tag*{(19.1)a}$$

may be expressed in terms of $h_{\lambda\mu}$ and $k_{\lambda\mu}$ as follows:

$$\begin{aligned} &U_{\omega}(\kappa\sqrt{|k|}\,k_{\mu\lambda} - \tfrac{1}{2}e_{\mu\lambda\alpha\beta}k^{\alpha\beta})\,(h^{\omega\mu}h^{\lambda\nu} + \frac{\kappa}{2}\sqrt{|k|}\,E^{\omega\mu\lambda\nu}) \\ &= \kappa(\nabla_{\lambda}\sqrt{|k|}\,k^{\lambda\nu}) + \tfrac{1}{2}E^{\omega\mu\lambda\nu}[(1-k)\nabla_{\omega}k_{\mu\lambda} + \sqrt{|k|}\,k_{\mu\lambda}(\nabla_{\omega}\sqrt{|k|})]. \end{aligned} \tag*{(19.1)b}$$

PROOF. We see from (16.3)a,b,c that (18.9)a is equivalent to

$$\kappa\partial_{[\omega}\sqrt{|k|}\,k_{\mu\lambda]} - \sigma\sqrt{g}\,U_{[\omega}f_{\mu\lambda]} - \tfrac{1}{3}e_{\omega\mu\lambda\nu}\nabla_{\alpha}k^{\alpha\nu} = 0$$

so that by I (2.12)d

$$2\nabla_{\alpha}k^{\alpha\nu} = E^{\omega\mu\lambda\nu}[\kappa\nabla_{\omega}\sqrt{|k|}\,k_{\mu\lambda} - \sigma\sqrt{g}\,U_{\omega}f_{\mu\lambda}]. \tag*{(19.2)}$$

---

[1] This equation is a consequence of (18.9)b.

[2] According to (18.10) we have $\mathfrak{w}^2 < 0$, whenever $\mathfrak{w}^2 \neq 0$.

[3] Schrödinger, E., (2); *Cf.* also Maurer-Tison, F., (2).

According to (18.9)b and (16.3)b a necessary and sufficient condition for (19.1)a may be expressed, according to II (7.1)c, as follows:

$$\sigma\sqrt{g}\,U_\lambda f^{\lambda\nu} = \kappa\,\nabla_\lambda \sqrt{|k|}\,k^{\lambda\nu} + \tfrac{1}{2}E^{\omega\mu\lambda\nu}\,\nabla_\omega\,k_{\mu\lambda}. \tag{19.3}$$

The equations (19.2) and (19.3) lead directly to (19.1)b.

THEOREM (19.2). *The system* (18.4)b *is equivalent to Maxwell's equations* (18.9) (*to the set* (3.2)a *of condensed Maxwell's equations*) *in the sense that every solution of* (18.4)b *is also a solution of* (18.9) (*of* (3.2)).

The proof follows the pattern of the proofs of Theorems (5.2) and (5.3).[1]

Preceding results justify the following

DEFINITION (19.1). *The system* (18.4)b *will be referred to as the generalized Maxwell set.*

20. Auxiliary results.

THEOREM (20.1). *The tensor* $^*A_{(\mu\lambda)}$ *defined by* IV (3.12)b *may be expressed in terms of* $f_{\lambda\mu}$ *and* w^ν *as follows:*

$$\begin{aligned} {}^*A_{(\mu\lambda)} = {}&{}^0B_{\mu\lambda} + B_{\mu\lambda\nu}\sqrt{G}\,w^\nu + B_{\mu\lambda\nu\omega}G w^\nu w^\omega \\ &+ \sigma(\nabla^\alpha\sqrt{G}\,w^\beta)e_{\beta\xi\eta\zeta}\,U^{\xi\eta\zeta}_{\alpha\lambda\mu} \end{aligned} \tag{20.1}$$

where the tensors ${}^0B_{\mu\lambda}$, $B_{\mu\lambda\nu}$, $B_{\mu\lambda\nu\omega}$ *have properties similar to those mentioned in Theorem* (6.1), *and* $U^{\xi\eta\zeta}_{\alpha\lambda\mu}$ *is to be taken from* (17.6).

The proof follows the pattern of the proof of Theorem (6.1).

In subsequent considerations we shall use the following abbreviations:

$$K^0{}_{\omega\mu\lambda} \overset{\text{def}}{=} {}^0K_{\omega\mu\lambda} - e_{\omega\mu\alpha\beta}\,\nabla_\lambda\,F^{\alpha\beta} \tag{20.2a}$$

$$\begin{aligned} \tfrac{1}{2}GI_{\mu\lambda} \overset{\text{def}}{=} {}&G(w_{(\mu}{}^*G_{\lambda)} - h_{\mu\lambda}w_\alpha{}^*G^\alpha + \tfrac{1}{2}w_\alpha G_{(\mu}f_{\lambda)}{}^\alpha) \\ &+ \tfrac{1}{4}(e_{\alpha\mu\xi\eta}\nabla^\beta F^{\xi\eta} + K^0{}_{\alpha\mu}{}^\beta)\,(e_{\beta\lambda\gamma\delta}\nabla^\alpha F^{\gamma\delta} + K^0{}_{\beta\lambda}{}^\alpha) \\ &+ \tfrac{1}{2}K^{0\alpha}{}_{(\mu}{}^\beta e_{\lambda)\eta\beta\alpha}w^\eta\sqrt{G} = \tfrac{1}{2}GI_{\lambda\mu} \\ &({}^*G_\lambda \overset{\text{def}}{=} G^\alpha f_{\alpha\lambda}) \end{aligned} \tag{20.2b}$$

---

[1] Instead of using II (7.25) we have to use the equation

$$\nabla_\alpha\,k_\omega{}^\alpha + 2U_\alpha k_\omega{}^\alpha + U_{\alpha\beta\omega}k^{\beta\alpha} = 0$$

derived from II (7.23) and (16.1)a.

$$(20.2)\text{c} \qquad \tfrac{1}{2}GQ_{\lambda\mu} \overset{\text{def}}{=} \tfrac{1}{4}K_{\alpha\mu}{}^{\beta}K_{\beta\lambda}{}^{\alpha} + {}^{*}A_{(\mu\lambda)}$$

$$- \tfrac{1}{8}(K_{\alpha\gamma\beta}K^{\beta\gamma\alpha} + 4{}^{*}A)h_{\mu\lambda}$$

$$-\frac{G}{2}\left\{\nabla^{\alpha}\left[\frac{1}{\sqrt{g}}(\kappa\sqrt{|k|}\,k_{\alpha\mu} - \tfrac{1}{2}e_{\alpha\mu\gamma\delta}k^{\gamma\delta})\right]\right\}$$

$$\cdot\left\{\nabla^{\beta}\left[\frac{1}{\sqrt{g}}(\kappa\sqrt{|k|}\,k_{\beta\lambda} - \tfrac{1}{2}e_{\beta\lambda\xi\eta}k^{\xi\eta})\right]\right\}$$

$$({}^{*}A \overset{\text{def}}{=} {}^{*}A_{\lambda\mu}h^{\lambda\mu}).$$

THEOREM (20.2). *The tensor $GQ_{\lambda\mu}$ may be expressed in terms of w^{ν} and $f_{\lambda\mu}$ as follows:*

$$(20.3)\text{a} \quad \frac{G}{2}Q_{\lambda\mu} = {}^{*}A_{(\mu\lambda)} + \frac{G}{2}(2w_{\alpha}\nabla_{(\mu}f_{\lambda)}{}^{\alpha} + I_{\mu\lambda}) + G\Lambda h_{\lambda\mu}$$

where

$$(20.3)\text{b} \qquad \Lambda \overset{\text{def}}{=} -\tfrac{1}{4}w^{2} - \frac{I}{4} - \tfrac{1}{2}{}^{*}\frac{A}{G}$$

$$(w^{2} \overset{\text{def}}{=} w^{\lambda}w_{\lambda},\ I \overset{\text{def}}{=} I_{\lambda\mu}h^{\lambda\mu})$$

and ${}^{*}A_{(\mu\lambda)}$ *and* $I_{\mu\lambda}$ *are to be taken from* (20.1) *and* (20.2)b.

PROOF. We find easily from (17.1)a,b and (20.2)a

$$(20.4) \qquad \sigma K_{\omega\mu\lambda} = \sqrt{G}w^{\nu}e_{\nu\omega\mu\lambda} + e_{\omega\mu\alpha\beta}\nabla_{\lambda}F^{\alpha\beta} + K^{0}{}_{\omega\mu\lambda}.$$

Therefore, one obtains, after some calculation,

$$(20.5)\text{a} \quad \tfrac{1}{4}K_{\alpha\mu\beta}K^{\beta\gamma\alpha}h_{\gamma\lambda} = \frac{G}{2}(w_{\mu}w_{\lambda} - w^{2}h_{\mu\lambda} + 2w_{\alpha}\nabla_{(\mu}f_{\lambda)}{}^{\alpha} + I_{\mu\lambda})$$

$$(20.5)\text{b} \qquad -\tfrac{1}{8}K_{\alpha\gamma\beta}K^{\beta\gamma\alpha} = \frac{G}{4}(w^{2} - I).$$

Substituting from these equations into (20.2)c, and remembering that according to (18.9)b and (16.3)b

$$(20.6) \qquad \sigma w_{\nu} = \nabla^{\alpha}\left\{\frac{1}{\sqrt{g}}[\kappa\sqrt{|k|}\,k_{\alpha\nu} - \tfrac{1}{2}e_{\alpha\nu\beta\gamma}k^{\beta\gamma}]\right\},$$

we obtain (20.3).

NOTE I. While (20.2)c defines $Q_{\mu\lambda}$ in a purely geometrical way, the same tensor is expressed by (20.3)a in terms of w^{ν}, $f_{\mu\lambda}$, and $h_{\lambda\mu}$.

We shall see later that even the latter tensor has a physical meaning.

NOTE II. An easy inspection shows that there is only one term in $\frac{1}{2}GI$ which contains w^2, namely

$$\tfrac{1}{4}e_{\alpha\mu\xi\eta}\,e_{\beta\lambda\gamma\delta}\,h^{\mu\lambda}(\nabla^\beta F^{\xi\eta})\,(\nabla^\alpha F^{\gamma\delta})$$
$$= -\frac{G}{2}(\nabla^\alpha f^{\beta\gamma})\,(\nabla_\alpha f_{\beta\gamma}) + Gw^2 + \dots .$$

Here we denote by ... other terms which do not contain w^2. Thus, Λ contains the term $-\frac{3}{4}w^2 + \frac{1}{4}(\nabla^\alpha f^{\beta\gamma})\,(\nabla_\alpha f_{\beta\gamma})$, as was to be expected according to (6.4)a.

21. Gravitational field equations.

THEOREM (21.1). *The equations* (16.1)a *and* (16.1)c *are equivalent* to (18.9)a and

(21.1)a $$H_{\lambda\mu} - \tfrac{1}{2}Hh_{\lambda\mu} = \tfrac{1}{2}GT_{\lambda\mu}$$

where

(21.1)b $$T_{\lambda\mu} = w_\lambda w_\mu + Q_{\lambda\mu},$$

and the second term in (21.1)b *has to be taken from* (20.3)a *and* (16.6)b.

The proof, based on results of the previous section, is similar to the proof of Theorem (8.1).

The form of (21.1) justifies the following

AGREEMENT (21.1). *We identify*

a) $h_{\lambda\mu}$ *with the tensor of the gravitational field,*
b) $GT_{\lambda\mu}$ *with the momentum energy tensor for matter and electromagnetism,*
c) G *with a constant multiple of the gravitational function,*
d) Λ *with a constant multiple of the cosmological function.*[1]

The equations (21.1) *will be referred to as gravitational field equations.*

NOTE I. Lichnerowicz, A., (3), suggested identifying the gravitational field with $^*h^{\lambda\nu}$ given by I (3.9)a. Its inverse tensor $^*\overset{-1}{h}_{\lambda\mu}$ given by IV (10.4) satisfies the requirements a), b) of Definition (8.1) but does not satisfy the requirement c) of this Definition:

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[1] See footnote 3 on page 182.

The system (8.8) does not admit a first quadratic integral

$$V^\lambda V^\mu \overset{-1}{{}^*h_{\lambda\mu}} = \text{const.}$$

Starting with the metric defined by ${}^*h^{\lambda\nu}\sqrt{-\,\mathfrak{g}}$ we obtain another expression for the momentum energy tensor (*Cf.* Hély, J., (2)). For another definition of a metric tensor, see Maurer-Tison, F., (1), (2) and Mavridès, S., (6).

NOTE II. We remind the reader that we have $G = g$ for the second class, while in the case of the third class $G = g = 1$.

A gravitational *function* instead of a gravitational constant appears also in Jordan, P., (1), (2); Hoffmann, B., (6), (4); Havlíček, F., (1); Nariai, H., (1); Heckmann, G. — Jordan, P. — Fricke, W., (1); Just, K., (1), (2), (3).

NOTE III. We shall see later the reasons which compel us to consider $GT_{\lambda\mu}$ instead of $T_{\lambda\mu}$ as momentum energy tensor.

NOTE IV. The remark about the continuity equation for matter (see Note I following Agreement (8.1)) is also appropriate here. It also justifies the name of momentum energy tensor *of matter* and electromagnetism.

If we use Definition (8.1) of a metric tensor, it is clear that Theorem (8.2) holds even here.

THEOREM (21.2). *The gravitational field and the electromagnetic field are interrelated.*[1] *The* $\left\{\begin{matrix}\textit{gravitational}\\ \textit{electromagnetic}\end{matrix}\right\}$ *field generates the* $\left.\begin{matrix}\textit{electromagnetic}\\ \textit{gravitational}\end{matrix}\right\}$ *field. These fields as well as the electric charge,*[2] *gravitational function, cosmological function, and momentum energy tensor spring from one basic conception, namely from the unified tensor field* $g_{\lambda\mu}$.

The proof based on previous results follows the pattern of the proof of Theorem (9.1).

The question arises whether our theory admits a tensor $g_{\lambda\mu}$

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[1] The idea of the interdependence to these two fields is not a new one. *Cf.* for instance, Nipher, F., (1), (2).

[2] If we adopt the continuity equation for matter, then the mass density also has this property.

leading to pure radiation. The next Theorem answers this question in the affirmative.

THEOREM (21.3). *The set of all tensors $g_{\lambda\mu}$ satisfying the equations of our theory and leading to pure radiation is not an empty set.*

PROOF. We shall exhibit at least one such tensor. We know that the vector $X_\lambda = \partial_\lambda X$ and the tensor IV (5.10) satisfy the equations of our theory. Since here $k = 0$, the equation (19.1)b reduces to

$$E^{\omega\mu\lambda\nu} \nabla_\omega k_{\mu\lambda} + e_{\mu}{}^{\nu}{}_{\alpha\beta} k^{\alpha\beta} U^\mu = 0,$$

and this equation is equivalent to

$$E^{\omega\mu\lambda\nu}(\nabla_\omega k_{\mu\lambda} - U_\omega k_{\mu\lambda}) = 0$$

or

$$\partial_{[\omega} \frac{k_{\mu\lambda]}}{\sqrt{g}} = 0. \tag{21.2}$$

An easy inspection of IV (5.10) shows that

$$g = 1/(1 - Ar^{-4})$$

so that

$$\partial_{[\omega} \frac{k_{\mu\lambda]}}{\sqrt{g}} = \delta_{[\mu}^{\mathrm{I}}{}_{\lambda}^{\mathrm{IV}}\, \partial_{\omega]}\, \varepsilon \sqrt{|BA| r^{-4}(1 - Ar^{-4})} = 0.$$

Thus, (21.2) is satisfied and, consequently, the tensor IV (5.10) leads to pure radiation.

ILLUSTRATION (21.1). It is not difficult to find the corresponding gravitational and electromagnetic fields from IV (5.10). We have

$$((h_{\lambda\mu})) = \left(\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & B(1 - Ar^{-4}) \end{pmatrix}\right). \tag{21.3}$$

The electromagnetic field results from IV (5.10) and (16.3)b

$$f^{\lambda\nu} = 2\delta^{[\lambda}_{\mathrm{II}}{}^{\nu]}_{\mathrm{III}} \frac{\varepsilon\sigma\sqrt{|A|}}{r^4 \sin\theta} \tag{21.4a}$$

$$f_{\lambda\nu} = 2\delta^{\mathrm{II}}_{[\lambda}{}^{\mathrm{III}}_{\nu]}\, \varepsilon\sigma\sqrt{|A|} \sin\theta. \tag{21.4b}$$

In particular,

$$f^{\lambda\nu}\sqrt{|\mathfrak{h}|} = 2\varepsilon\sigma r^{-2}\sqrt{|AB|(1 - Ar^{-4})}\,\delta^{\lambda}_{[\mathrm{II}}{}^{\nu}_{\mathrm{III}]}.$$

This last equation leads easily to

$$\partial_\lambda f^{\lambda\nu}\sqrt{|\mathfrak{h}|} = 0,$$

while (21.4)b yields

$$\partial_{[\omega} f_{\mu\lambda]} = 0.$$

The gravitational function results from (12.3) and (21.4)

$$G = 1/(1 - Ar^{-4}).$$

Hence, G is constant at infinity: $r \to \infty$.

NOTE. After having interpreted in the most general case the field equations (16.1) in terms of physical quantities, we may elaborate on Note IV following Theorem IV (4.2). We shall explain why the identification of the electromagnetic field with $R_{[\omega\mu]}$ is not a suitable one. This identification is obviously suggested by (16.1)b, which is equivalent to the first set of Maxwell's equations for the electromagnetic field $R_{[\omega\mu]}$. In order to find the second set of Maxwell's equations we have to start with the Definition of the current density vector

$$w^\nu\sqrt{|\mathfrak{h}|} = \partial_\lambda R^{[\lambda\nu]}\sqrt{|\mathfrak{h}|}. \tag{21.5}$$

This set could be converted into the second set of Maxwell's equations provided that our remaining field equations (16.1)a and (16.1)c yield another condition for w^ν or $R_{[\lambda\mu]}$. This is an impossible requirement, for neither S_λ nor $R_{(\lambda\mu)}$ can be expressed in terms of w^ν or $R_{[\lambda\mu]}$. Thus, if we identify $R_{[\omega\mu]}$ with the electromagnetic field, *we are unable to derive the second set of Maxwell's equations from our field equations.* The situation, may be described by saying that the field equations (16.1)a and (16.1)c do not express any property of the electromagnetic field $R_{[\omega\mu]}$. These reasons prevented us from using the above identification.

22. Momentum energy tensor. Motion.

THEOREM (22.1). *The momentum energy tensor $GT^{\lambda\nu}$ satisfies the following conditions:*

$$\nabla_\lambda GT^{\lambda\nu} = 0. \tag{22.1}$$

(*Law of conservation of momentum and energy.*)

PROOF. The equation (22.1) is a consequence of IV (3.4)e and (21.1)a.

NOTE I. The equation (22.1) shows the reason why we have to consider $GT_{\lambda\mu}$ instead of $T_{\lambda\mu}$ as a momentum energy tensor. If G were a constant, it would be irrelevant whether we used $T_{\lambda\mu}$ or $GT_{\lambda\mu}$.

NOTE. The equations (22.1) are substantially the same as the equations obtained by Bose, S. N., (1).

In the next Theorem we use the expression $\mathscr{H}$ defined by (10.2).

THEOREM (22.2). *Put*

(22.2)a $$G\mathfrak{T}_\lambda{}^\nu \overset{\text{def}}{=} GT_\lambda{}^\nu\sqrt{|\mathfrak{h}|}$$

(22.2)b $$G\mathfrak{t}_\lambda{}^\nu \overset{\text{def}}{=} \left(\delta_\lambda^\nu \mathscr{H} - \frac{\partial \mathscr{H}}{\partial\partial_\nu h^{\alpha\beta}}\,\partial_\lambda h^{\alpha\beta}\right).$$

Then (22.1) *is equivalent to*

(22.3) $$\partial_\nu G(\mathfrak{T}_\lambda{}^\nu + \mathfrak{t}_\lambda{}^\nu) = 0.$$

The proof follows the pattern of the proof of Theorem (10.2).

THEOREM (22.3). *The equations of motion are*

(22.4) $$w^\lambda \nabla_\lambda G w^\nu + \nabla_\lambda G Q^{\lambda\nu} = 0.$$

PROOF. The equations (22.4) result from (22.1), (21.1)b and (16.9)b.

NOTE. Here we could repeat the first two objections discussed in section 11. Of course, these objections are now to be directed against (22.4).

23. Unified law of inertia. In order to dispose of the above objections we shall introduce the Agreement (12.1) which we restate here.

AGREEMENT (23.1). *Henceforth in this part we replace the field equations* (16.1)b *with*

(23.1)a $$D_\lambda G Q^{\lambda\nu} = 2U^\nu{}_{\lambda\mu}H^{\lambda\mu} + S_\alpha G Q^{\alpha\nu} + U_\alpha(GQ^{\alpha\nu} + 2Hh^{\alpha\nu}).$$

NOTE I. The equations (23.1)a reduce by virtue of (16.1)a to

(23.1)b $$D_\lambda G Q^{\lambda\nu} = 2U^\nu{}_{\lambda\mu} H^{\lambda\mu} + U_\alpha(GQ^{\alpha\nu} + 2Hh^{\alpha\nu}).$$

NOTE II. The equations II (3.1) and (20.2)c show that (23.1) may be looked upon as purely geometrical conditions. On the other hand, the equations (17.9) and (20.3) enable us to convert these purely geometrical conditions into physical ones.

NOTE III. We shall not discuss here the slight changes (in formulating the previous results) due to Agreement (23.1). The reader can easily follow the pattern of reappraisal exhibited in section 12. Let us stress only that (23.1)b in its physical form leads again from (16.6)b to (18.9)b. Moreover, we again obtain Theorems (21.1), (21.2), (22.1) and (22.2). However, Theorem (22.3) has to be changed although (22.4) holds.

THEOREM (23.1)a. *The equations of motion are*

(23.2)a $$w^\lambda D_\lambda G w^\nu = 0.$$

PROOF. According to II (7.1) we have

(23.3) $$O = 6U_{(\nu\lambda\mu)} h^{\lambda\mu} = 2U_{\nu\lambda\mu} h^{\lambda\mu} + 4U_\nu.$$

Moreover, by virtue of (16.1)a

(23.4)a $$D_\lambda G Q^{\lambda\nu} = \nabla_\lambda G Q^{\lambda\nu} + U_\alpha G Q^{\alpha\nu} + U^\nu{}_{\lambda\mu} G Q^{\lambda\mu},$$

(23.4)b $$w^\lambda D_\lambda G w^\nu = w^\lambda \nabla_\lambda G w^\nu + U^\nu{}_{\mu\lambda} G w^\mu w^\lambda,$$

so that according to (21.1), (22.4) and (23.3)

(23.5) $$\begin{aligned} D_\lambda G Q^{\lambda\nu} + w^\lambda D_\lambda G w^\nu &= 2U^\nu{}_{\mu\lambda}(H^{\mu\lambda} - \tfrac{1}{2} H h^{\mu\lambda}) + U_\alpha G Q^{\alpha\nu} \\ &= 2U^\nu{}_{\mu\lambda} H^{\mu\lambda} + U_\alpha(GQ^{\alpha\nu} + 2Hh^{\alpha\nu}). \end{aligned}$$

This equation reduces to (23.2)a by virtue of (23.1)b.

NOTE. Denote by

(23.6) $$w^\nu \overset{\text{def}}{=} \frac{dx^\nu}{dp}$$

the vector involved in (23.2)a and by $x^\nu = x^\nu(p)$ the solution of (23.6). Then $G = G(p)$ may be considered as a function of p.

THEOREM (23.1)b. *Put*

$$q = c_1 \int \frac{dp}{G} + c_2, \quad (c_1,\ c_2 = \text{const},\ c_1 \neq 0). \tag{23.7}$$

Then (23.2)a *is equivalent to*

$$\frac{d^2x^\nu}{dq^2} + \Gamma_\lambda{}^\nu{}_\mu \frac{dx^\lambda}{dq}\frac{dx^\mu}{dq} = 0. \tag{23.2)b}$$

This equation admits the first integral

$$h_{\lambda\mu}\frac{dx^\lambda}{dq}\frac{dx^\mu}{dq} = c^2\ (= \text{const}). \tag{23.8}$$

PROOF. An easy inspection shows that (23.2)a may be written

$$\begin{aligned} w^\lambda D_\lambda G w^\nu &= \frac{d}{dp}\left(G\frac{dx^\nu}{dp}\right) + \Gamma_\lambda{}^\nu{}_\mu G \frac{dx^\lambda}{dp}\frac{dx^\mu}{dp} \\ &= G\left(\frac{dq}{dp}\right)^2\left[\frac{d^2x^\nu}{dq^2} + \Gamma_\lambda{}^\nu{}_\mu \frac{dx^\lambda}{dq}\frac{dx^\mu}{dq}\right] \\ &+ \frac{dq}{dp}\frac{dx^\nu}{dp} G\left[\frac{d\ln G}{dq} + \frac{d}{dq}\ln\frac{dq}{dp}\right] = 0. \end{aligned}$$

This equation leads immediately, because of (23.7), to (23.2)b. The remaining statement of our Theorem follows from Theorem (8.2).

THEOREM (23.2). (*The unified law of inertia*). *A charged particle moving freely in the unified field*

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} = h_{\lambda\mu} + \frac{\sigma}{2} e_{\lambda\mu\alpha\beta}F^{\alpha\beta} + \sigma L_{\lambda\mu}{}^{1} \tag{23.9}$$

describes a path (*an autoparallel line*) *of the unified connection* $\Gamma_\lambda{}^\nu{}_\mu$ *given by* II (2.1).

PROOF. This Theorem is a verbal description of Theorems (23.1)a,b.

NOTE. The above results dispose of the objections mentioned earlier. In order to show that our theory may be looked upon as a generalization of classical results, we have to prove that the unified law of inertia may be brought into a form which expresses Newtonian laws in an appropriate approximation.

[1] See (16.3)c.

24. Newtonian laws. This section is based on the results of section 14 and on (23.2)b. In order to be able to use the results of section 14 without changing notation, we denote in this section by v^ν the vector $\dfrac{dx^\nu}{dq}$:

$$v^\nu \overset{\text{def}}{=} \frac{dx^\nu}{dq} \tag{24.1}$$

so that (23.2)b may be rewritten in the form (13.1), *i.e.*,

$$v^\lambda D_\lambda v^\nu = \frac{d^2x^\nu}{dq^2} + \Gamma_\lambda{}^\nu{}_\mu \frac{dx^\lambda}{dq}\,\frac{dx^\mu}{dq} = 0. \tag{24.2}$$

Then we may use here all definitions and results of section 14 except possibly Lemmas (14.3) and (14.4). In regard to the tensor $U_{\nu\lambda\mu}$ involved in these Lemmas, we see from II (7.2), Theorem II (7.2) and Corollary II (7.1) that the use of (14.6) leads to (14.7) even here.[1] Hence Lemma (14.3) holds, and, therefore, the same is true for Lemma (14.4). Consequently, all results and definitions of section 14 may be used here also.

THEOREM (24.1). *There is always an admissible coordinate system x^ν for which*

$$\partial_{\mathrm{IV}} h_{\lambda\,\mathrm{IV}} \overset{0}{=} 0 \tag{24.3a}$$

$$\begin{aligned}\partial_\xi(v^\lambda \nabla_\lambda v^\nu) + \mathscr{U}^\nu{}_{\omega\mu\xi}\, v^\omega v^\mu + (\partial_\xi k^{\beta\alpha})\,(\partial_\beta k_\alpha{}^\nu)\\ + \tfrac{1}{2}(\partial^\nu k_{\alpha\beta})\,(\partial_\xi k^{\alpha\beta}) \overset{0}{=} 0.\end{aligned} \tag{24.3b}$$

The proof follows the pattern of the proof of Theorem (15.1).

DEFINITION (24.1). *Every admissible coordinate system for which* (24.3) *holds will be termed an allowable system for the first two classes.*

THEOREM (24.2). *Every allowable coordinate system x^ν for the first two classes has the following properties:*

a) *The unified law of inertia* (23.2)b *expressed in x^ν reduces at O to the classical Newtonian law of gravitation.*

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[1] Although we have, by virtue of I (6.2), II (9.2) and (14.1) $\underset{i}{\lambda} \overset{0}{=} 0$, $\underset{ijk}{\lambda} \overset{0}{=} 1$ we cannot, at O, use II (11.7) in conjunction with II (10.6); for whenever (14.1)b holds the nonholonomic frame involved in II (10.6)b is not defined at O.

b) *The unified law of inertia* (23.2)b *expressed in* x^ν *reduces in the linear neighborhood of* O *to the classical relativistic second law of Newton for the classical relativistic electromagnetic force vector* F^ν (*defined by* (15.8)).

PROOF. First of all, we have according to (14.1)

$$\nabla_\lambda k^{\lambda\nu} \overset{0}{=} \partial_\lambda k^{\lambda\nu} \tag{24.4a}$$

$$\nabla_\beta k^{\beta\alpha} k_\alpha{}^\nu \overset{1}{=} x^\xi[(\partial_\beta k^{\beta\alpha})\partial_\xi k_\alpha{}^\nu + (\partial_\beta k_\alpha{}^\nu)\partial_\xi k^{\beta\alpha}] \tag{24.4b}$$

$$\tfrac{1}{4}\nabla^\nu k^{\beta\alpha} k_{\beta\alpha} \overset{1}{=} \tfrac{1}{2}x^\xi(\partial^\nu k_{\beta\alpha})\,(\partial_\xi k^{\beta\alpha}). \tag{24.4c}$$

Moreover, by virtue of (16.3) and (14.1)

$$f_{\lambda\nu} \overset{0}{=} 0, \quad f_{\lambda\nu} \overset{1}{=} -\frac{\sigma}{2}\,{}^0e_{\lambda\nu\alpha\beta} x^\xi \partial_\xi k^{\alpha\beta} \tag{24.5}$$

where

$${}^0e_{\lambda\nu\alpha\beta} = \mathfrak{e}_{\lambda\nu\alpha\beta}(\sqrt{|\mathfrak{h}|})_O.$$

Hence, the first set of Maxwell's equations yields

$$\partial_\lambda k^{\lambda\nu} \overset{0}{=} 0. \tag{24.6}$$

On the other hand, we obtain from (16.3), (15.8) and (15.9)

$$\frac{g}{1-k} M_\lambda{}^\nu = k^{\nu\alpha} k_{\alpha\lambda} + \tfrac{1}{4}\delta_\lambda^\nu k^{\alpha\beta} k_{\alpha\beta} \tag{24.7a}$$

$$F^\nu = (\nabla_\beta k^{\beta\alpha} k_\alpha{}^\nu + \tfrac{1}{4}\nabla^\nu k^{\alpha\beta} k_{\alpha\beta})\frac{1-k}{g} + M^{\lambda\nu}\partial_\lambda \ln\frac{1-k}{g} \tag{24.7b}$$

so that according to (24.4) and (24.6)

$$F^\nu \overset{1}{=} x^\xi[(\partial_\xi k^{\beta\alpha})\partial_\beta k_\alpha{}^\nu + \tfrac{1}{2}(\partial^\nu k_{\alpha\beta})\partial_\xi k^{\alpha\beta}]_O. \tag{24.8}$$

From this point on the proof follows the pattern of the proof of Theorem (15.2). [1]

25. Special cases.

In this concluding section we shall consider some special cases discussed in the second chapter.

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[1] For another approach to the same problem, based on Einstein, A. — Infeld, L. — Hoffmann, B., (1), see Infeld, L., (1), (2), (3), (4); Callaway, J., (1).

1) We know that if II (6.4) is satisfied, then according to Theorem II(6.1) (for $n = 4$)

$$S_{\omega\mu\nu} = \tfrac{2}{3} S_{[\omega} h_{\mu]\nu}. \tag{25.1}$$

Therefore, if II (6.4) holds, the field equation (16.1)a annihilates $S_{\omega\mu}{}^{\nu}$, and according to II (3.2), also $U^{\nu}{}_{\lambda\mu}$:

$$S_{\omega\mu}{}^{\nu} = 0, \quad U^{\nu}{}_{\lambda\mu} = 0. \tag{25.2}$$

Therefore, II (3.1) reduces to

$$\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \{\lambda{}^{\nu}{}_{\mu}\}. \tag{25.3}$$

Hence,

$$R_{\mu\lambda} = H_{\mu\lambda} \tag{25.4}$$

so that (16.1)b is automatically satisfied by $X_\lambda = \partial_\lambda X$, and (16.1)c reduces to

$$H_{\mu\lambda} = 0. \tag{25.5}$$

The condition II (2.1)b reduces to

$$D_\omega g_{\lambda\mu} = \nabla_\omega g_{\lambda\mu} = \nabla_\omega k_{\lambda\mu} = 0 \tag{25.6}$$

so that, in particular,

$$\partial_{[\omega} k_{\mu\lambda]} = 0, \quad \nabla_\alpha k^{\alpha\nu} = 0. \tag{25.7}$$

Hence, $k_{\mu\lambda}$ may be identified with the tensor of the electromagnetic field. Then, of course, we have a case of pure radiation. If $k_{\lambda\mu}$ belongs to the first class, then according to (25.2) and Theorem III (6.1), every pair of basic vectors is X_2-forming and every basic value (eigenvalue) $\underset{i}{\lambda}$ is constant.

2) If II (7.4)a holds, then we have according to Corollary II (7.1)

$$U^{\nu}{}_{\lambda\mu} = 0, \tag{25.8}$$

and the equations (23.2)b reduce to

$$\frac{d^2x^\nu}{dq^2} + \{\lambda{}^{\nu}{}_{\mu}\} \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = 0. \tag{25.9}$$

Hence, every path of $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is at the same time a geodesic, and vice versa.[1] The unified law of inertia is identical with the law of inertia in the gravitational theory. From (25.8), (16.1)a and II (7.23) we have

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[1] See Theorem II (7.4).

$$(25.10) \qquad \nabla_\alpha k^{\alpha\nu} = 0.$$

Moreover, we know according to Corollary II (7.2) that g and K are constant. Hence, we see from

$$g = 1 + 2K + k$$

that k is also constant. Thus, we obtain from (16.3)a, (25.10), (16.1)a and Theorem IV (4.1)

$$(25.11) \qquad 0 = \partial_{[\mu} f_{\lambda\nu]} = \frac{\sigma\kappa}{\sqrt{g}} \sqrt{|k|}\, \partial_{[\mu} k_{\lambda\nu]}.$$

Hence, if $S_{[\mu\lambda\nu]} \neq 0$ [1], we see from II (6.12) and (25.11) that $k = 0$. Therefore, $k_{\lambda\mu}$ does not belong to the first class. This particular case was considered in detail in Hlavatý, V., (14).

3) If II (7.22)a holds, then we have according to Theorem II (7.6)a for $n = 4$

$$(25.12) \qquad -3U^\nu{}_{\lambda\mu} = 2U^\nu h_{\lambda\mu} - U_\mu \delta^\nu_\lambda - U_\lambda \delta^\nu_\mu.$$

Therefore, the unified law of inertia (23.2)b reduces to

$$\frac{d^2x^\nu}{dq^2} + \{\lambda{}^\nu{}_\mu\} \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = \tfrac{2}{3} U^\nu c^2 - \tfrac{2}{3} U_\mu \frac{dx^\mu}{dq} \frac{dx^\nu}{dq},$$

where c^2 is the constant involved in (23.8). This equation is equivalent to

$$(25.13) \qquad \frac{dx^{[\omega}}{dq} \left(\frac{d^2x^{\nu]}}{dq^2} + \{\lambda{}^{\nu]}{}_\mu\} \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} \right) = \tfrac{2}{3} c^2 \frac{dx^{[\omega}}{dq} U^{\nu]}.$$

Hence, paths of photons (*i.e.*, $c^2 = 0$) are the same as in gravitational theory, while every charged particle moving with a velocity different from the velocity of light (*i.e.*, $c^2 \neq 0$) describes a path different from a geodesic. [2] From (25.12), II (7.1)c, II (7.23) and (16.1)a we have

$$(25.14) \qquad \sqrt{g}[\nabla_\alpha k_\omega{}^\alpha + U_\alpha k_\omega{}^\alpha] = \nabla_\alpha \sqrt{g}\, k_\omega{}^\alpha = 0.$$

---

[1] A connection with $U^\nu{}_{\lambda\mu} = 0$, $S_{[\mu\lambda\nu]} = 0$, $S_\lambda = 0$ was introduced by Eisenhart, L. P., (3).

[2] Provided that $U^{[\nu} \dfrac{dx^{\omega]}}{dq} \neq 0$. See also Theorem II (7.5).

Therefore, the field $m_{\lambda\nu}$ defined by IV (4.16) satisfies the condition

$$\partial_{[\omega} m_{\mu\lambda]} = 0$$

equivalent to (16.1)a. This field was identified with electromagnetic field in Hlavatý, V., (18). In that paper the above situation was discussed in more detail and applied to celestial mechanics under the assumption that $\nabla_\lambda k_{\mu\nu}$ (but not necessarily $k_{\mu\nu}$) is a weak field.[1] It turned out that the predictions for the deflection of light and for the shift of spectral lines are the same as in the gravitational theory, while the prediction for the perihelion of Mercury yields a different result than in gravitational theory. Hence, in this theory the prediction of the latter phenomenon is no longer linked to the two former predictions.[2]

This fact could be of some importance in the future, should a more accurate measurement reveal some deviation of the world-line of Mercury from a geodesic line.

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[1] This conception is explained in Definition IV (1.1) of the above paper.

[2] In gravitational theory the predictions of all three phenomena are controlled by the same rule, namely that the corresponding trajectories are geodesics.

CHAPTER VI

APPENDIX I

1. Geometry of a light beam. [1] This section is devoted to the elementary geometry of a light beam from the point of view of unified field theory. It is closely connected with the elementary geometric background of the Michelson-Morley experiment.

In the book of P. G. Bergmann (1) (p. 133) we find the following lines:

"The experiment of Michelson and Morley has been repeated many times under varying conditions. All these new experiments have confirmed the original results, with the exception of those which were carried out by D. C. Miller. It is very difficult to decide why Miller's experiments show an effect which appears to indicate an "ether drift" of about 10 km/sec. However, since all the evidence of other experiments points to the accuracy of the Lorentz transformation equations, it is reasonable to assume that Miller's results were caused by a systematic experimental error which has not yet been discovered."

Far from being able to discover this error, [2] we wish to investigate here the most elementary geometric idea which from the point of view of the unified field theory could imply a positive effect in experiments of this type.

Denote by $t_{(a)}$ $(a = 1, 2)$ the time needed by the light signal sent from a source O to reach a point $P_{(a)}$. We shall investigate the difference $t_{(1)} - t_{(2)}$ under the following assumptions:

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[1] In this section the Greek (Roman) indices run from 1 to 4 (from 1 to 3). Moreover, very often we shall write the letters x, y, z, t for x^1, x^2, x^3, x^4 and consider x, y, z as orthogonal coordinates in our three-dimensional Euclidean space E_3.

[2] Shankland, R. S. — McCuskey, S. W. — Leone, F. C. — Kuerti, G., (1) subjected Miller's observations to a new analysis. They summarized their results as follows: "The small periodic fringe displacements found by Miller are due in part to statistical fluctuations in the readings of the fringe positions in a very difficult experiment. The remaining systematic effects are ascribed to local temperature conditions .... Miller's extensive Mount Wilson data contain no effect of the kind predicted by the aether theory, and, within the limitations imposed by local disturbances, are entirely consistent with a null result at all epochs during a year."

1) The points O, $P_{(1)}$, $P_{(2)}$ are linearly independent and

$$\overline{OP}_{(1)} = \overline{OP}_{(2)}. \tag{1.1}$$

2) *The velocity of light is constant.* We shall denote it by 1.[1]

3) The gravitational part of the unified field is so weak that it may be neglected.[2]

Using assumption 3) we may put

$$((h_{\lambda\mu})) = \left(\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right). \tag{1.2}$$

In subsequent considerations we shall denote this tensor by $a_{\lambda\mu}$, as is usual in differential geometry:

$$a_{\lambda\mu} \overset{\text{def}}{=} h_{\lambda\mu}, \tag{1.3}$$

so that

$$(dx)^2 + (dy)^2 + (dz)^2 - (dt^2) = a_{\lambda\mu} dx^\lambda dx^\mu. \tag{1.4}$$

According to assumption 2), we have for the tangential vector $\frac{dx^\nu}{dq}$ of a beam of light

$$a_{\lambda\mu} \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = 0 \tag{1.5)a}$$

$$\left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2 + \left(\frac{dz}{dq}\right)^2 = \left(\frac{dt}{dq}\right)^2 \neq 0. \tag{1.5)b}$$

(1.5) is a first integral of the law of inertia V (23.2)b for a beam of light. The latter equation can be rewritten, because of (1.2),

$$\frac{d^2x^\nu}{dq^2} + U^\nu{}_{\lambda\mu} \frac{dx^\lambda}{dq} \frac{dx^\mu}{dq} = 0. \tag{1.6}$$

---

[1] Velocity of light in connection with the Born theory is discussed in Hittmair, O. — Schrödinger, E., (1).

[2] Unified fields exist without gravitation. (*Cf.* for instance, Theorem V (9.2)). It is usual to ignore possible gravitational effects when discussing the Michelson-Morley experiment, and, in here neglecting the gravitational field, we follow this practice.

According to (1.5)b, a solution of (1.6) may be looked upon as defining a nonminimal curve C in our three-dimensional Euclidean space E_3 with the element of arc length dt. In order to derive the differential equations of C in E_3, we have to eliminate the parameter q from (1.6) by means of the last of these equations

$$\frac{d^2t}{dq^2} + U^4{}_{\lambda\mu}\frac{dx^\lambda}{dq}\,\frac{dx^\mu}{dq} = 0. \tag{1.7}$$

To this effect we first introduce the vector

$$v^\nu \overset{\text{def}}{=} \frac{dx^\nu}{dt}. \tag{1.8}$$

Then the result of elimination is

$$\dot{v}^a - \Omega^a{}_{\lambda\mu}v^\lambda v^\mu = 0 \tag{1.9}$$

where $\dot{v}^a = \dfrac{d}{dt}v^a$ and

$$\Omega^\nu{}_{\lambda\mu} \overset{\text{def}}{=} 2v^{[\nu}U^{4]}{}_{\lambda\mu}. \tag{1.10}$$

The corresponding first integral of (1.9) may be written according to (1.5)b

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 1 = a_{bc}v^bv^c. \tag{1.11}$$

Hence, v^a is the tangent unit vector of C. From (1.10) we have $\Omega^4{}_{\lambda\mu} = 0$, so that, in particular, by virtue of (1.9)

$$\begin{aligned} a_{bc}\dot{v}^bv^c &= a_{bc}\Omega^b{}_{\lambda\mu}v^\lambda v^\mu v^c \\ &= a_{\nu\omega}v^\nu v^\omega U^4{}_{\lambda\mu}v^\lambda v^\mu - a_{\nu\omega}U^\nu{}_{\lambda\mu}v^\lambda v^\mu v^\omega = 0\,^1. \end{aligned} \tag{1.12}$$

Hence, $\dot{v}^b$ either vanishes or is perpendicular to v^b, as was to be expected. Its length in E_3 is obviously the first curvature k of C:

$$\begin{aligned} k^2 &= a_{bc}\dot{v}^b\dot{v}^c = a_{bc}\Omega^b{}_{\lambda\mu}v^\lambda v^\mu\dot{v}^c \\ &= a_{\nu\omega}\Omega^\nu{}_{\lambda\mu}v^\lambda v^\mu\dot{v}^\omega \\ &= v^\nu\dot{v}^\omega a_{\nu\omega}U^4{}_{\lambda\mu}v^\lambda v^\mu - U_{\omega\lambda\mu}v^\lambda v^\mu\dot{v}^\omega \\ &= -U_{\nu\lambda\mu}v^\lambda v^\mu\dot{v}^\nu.\,^2 \end{aligned} \tag{1.13}$$

---

[1] We have
$$a_{\nu\omega}v^\nu v^\omega = 0$$
$$U_{\omega\lambda\mu}v^\lambda v^\mu v^\omega = U_{(\omega\lambda\mu)}v^\lambda v^\mu v^\omega = 0.$$
(*Cf.* II (7.1)b.)

[2] We have
$$a_{\nu\omega}v^\nu\dot{v}^\omega = a_{bc}v^b\dot{v}^c = 0.$$

Hence, a necessary and sufficient condition for C to be a straight line is

$$U_{\nu\lambda\mu} v^\lambda v^\mu \dot{v}^\nu = 0. \tag{1.14}$$

This equation is satisfied if, for instance,

$$U^\nu{}_{\lambda\mu} = 2p^\nu h_{\lambda\mu} - \delta^\nu_\lambda p_\mu - \delta^\nu_\mu p_\lambda.^{1} \tag{1.15}$$

In any case, if we denote by $\delta_{(a)}$ the length of $\overline{OP}_{(a)}$ we have for a sufficiently small $\delta_{(a)}$ [2]

$$\delta_{(a)}{}^2 = t_{(a)}{}^2 - \frac{t_{(a)}{}^4}{4!} 2(k_{(a)}{}^2)_O \tag*{(1.16)a}$$

where $(k_{(a)}{}^2)_O$ is the value at O of the square of the first curvature

$$(k_{(a)}{}^2)_O = (- U_{\nu\lambda\mu} v^\lambda{}_{(a)} v^\mu{}_{(a)} \dot{v}^\nu{}_{(a)})_O \tag{1.17}$$

and $v^\nu{}_{(a)}$ is the unit tangential vector of the beam joining O and $P_{(a)}$. If we disregard terms containing p-th powers of t, $p > 3$, then we may write instead of (1.16)a

$$\delta_{(a)} = t_{(a)} \left(1 - \frac{t_{(a)}{}^2 (k_{(a)}{}^2)_O}{4!}\right) \tag*{(1.16)b}$$

so that according to (1.1)

$$4!(t_{(1)} - t_{(2)}) = t_{(1)}{}^3 (k_{(1)}{}^2)_O - t_{(2)}{}^3 (k_{(2)}{}^2)_O. \tag{1.18}$$

When (1.14) holds, for instance when (1.15) holds, we obtain, instead of (1.18),

$$t_{(1)} = t_{(2)}. \tag{1.19}$$

The formula (1.18) may be of particular interest as shown by the following remarks:

(1) (1.18) may be considered as a theoretical basis for an experimental test of the unified field theory. The only theoretical complication is the computation of $U_{\nu\lambda\mu}$ from the given data. [3]

(2) Even in the presence of some electromagnetic fields the formula (1.18) might reduce to (1.19). Then, of course, we would expect Michelson and Morley's result.

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[1] *Cf.* Theorem II (7.5).

[2] *Cf.* Hlavatý, V., (1), p. 47.

[3] *Cf.* Theorems II (11.3), II (18.1). However, one could also use the approximation given by II (8.6)a.

(3) For other fields the formula (1.18) does not reduce to (1.19), in spite of the fact that it has been derived under the assumption that *the velocity of light propagation is constant.* One cannot help thinking of Miller's result.

Hence, from the point of view of the unified field theory and under the assumption of the constant velocity of light, one has to expect either Michelson-Morley's result or Miller's result. In other words, *from the point of view of the unified field theory, Miller's result, properly interpreted, is not necessarily at variance with the assumption of the constant velocity of light propagation.*

APPENDIX II

FOUR-COMPONENT SPINOR ALGEBRA AND ANALYSIS OF THE UNIFIED FIELD $g_{\lambda\mu}$.[1]

I. *Local spinor space* $\mathcal{S}_3(O)$

1. Prerequisites. 1) In this appendix we shall be compelled to make use of the following different types of indices:

a) small Greek indices α, β, ... running from I to IV,
b) small Roman indices a, b, ... running from 1 to 4,
c) small Roman indices a, b, ... running from *1* to *4*,
d) small Gothic indices $\mathfrak{a}$, $\mathfrak{b}$, ... running from $1^{\cdot}$ to $2^{\cdot}$,
e) capital Greek indices Λ, Ω, ... running from I to VI,
f) capital Roman indices A, B, ... running from 1 to 6.

With the exception of x, y, z, t, x, y, z, t all indices follow the usual summation convention.

Some knowledge of the elements of algebraic line geometry will be required. The reader not familiar with it will find all necessary conceptions of line geometry in the last appendix.

Throughout this appendix we confine ourselves to the *first* class.

2) Denote by O an arbitrary point of our four-dimensional space-time, by $T_4(O)$ the four-dimensional tangent space at O, and by $P_3(O)$ the ideal three-dimensional space of $T_4(O)$.[2] The space $P_3(O)$ is obviously a projective space. It contains the quadric H whose coordinates are given by $\rho h_{\lambda\mu}$, $(\rho \neq 0)$.

Let v^ν be an arbitrary vector in $T_4(O)$. In addition to $v^{\mathrm{I}}, \ldots, v^{\mathrm{IV}}$ we introduce the *redundant* components

$$v^{\mathrm{V}} = v^{\mathrm{VI}} = (v^\lambda v^\mu h_{\lambda\mu})^{\frac{1}{2}}. \tag{1.1}$$[3]

The set of all components $v^{\mathrm{I}}, \ldots, v^{\mathrm{VI}}$ will be symbolized by v^Ω.[4] Thus, for instance, we have for the basic null vectors $\underset{i}{a}^\nu$ according to (1.1)

---

[1] This appendix is based on the following papers: Cartan, E., (1), Veblen, O., (1), Hlavatý, V., (2), (3), (9).

[2] *i.e.*, the three-dimensional space "at infinity".

[3] The advantages in using these redundant components will be seen later.

[4] v^Ω are homogeneous coordinates. See footnote 2 on page 229.

$$\underset{i}{a}^{\Omega} = \delta^{\Omega}_{\lambda}\underset{i}{a}^{\lambda}. \tag*{(1.2)}$$

Besides the vectors $\underset{i}{a}^{\Omega}$, we introduce the following set of symbols

$$\underset{5}{a}^{\mathrm{V}} = -\underset{6}{a}^{\mathrm{VI}} = -\epsilon\sqrt{2}, \quad \epsilon = \pm 1, \tag*{(1.3)}$$

$$\text{the remaining } \underset{5}{a}^{\Omega} = \underset{6}{a}^{\Omega} = 0.$$

The set $\underset{1}{a}^{\Omega}, \ldots, \underset{6}{a}^{\Omega}$ will be symbolized by $\underset{A}{a}^{\Omega}$. The inverse set $\overset{A}{a}_{\Omega}$ is defined by

$$\overset{A}{a}_{\Omega}\underset{B}{a}^{\Omega} = \delta^{A}_{B}, \quad \overset{A}{a}_{\Omega}\underset{A}{a}^{\Lambda} = \delta^{\Lambda}_{\Omega}. \tag*{(1.4)}$$

We have, in particular,

$$\overset{i}{a}_{\Omega} = \delta^{\lambda}_{\Omega}\overset{i}{a}_{\lambda} \tag*{(1.5)}$$

$$\overset{5}{a}_{\mathrm{V}} = -\overset{6}{a}_{\mathrm{VI}} = -\epsilon/\sqrt{2}$$

$$\text{the remaining } \overset{5}{a}_{\Omega} = \overset{6}{a}_{\Omega} = 0.$$

The sets $\underset{A}{a}^{\Omega}$ and $\overset{A}{a}_{\Omega}$ will be referred to as a *redundant nonholonomic frame of reference.* This frame of reference gives rise to *nonholonomic redundant components* v^{A} of a vector

$$v^{A} = \overset{A}{a}_{\Omega}v^{\Omega}, \quad v^{\Omega} = v^{A}\underset{A}{a}^{\Omega}. \tag*{(1.6)a}$$

We have, in particular,

$$v^{5} = -v^{6} = -\epsilon\sqrt{\omega} \tag*{(1.6)b}$$

$$\omega \overset{\text{def}}{=} v^{\lambda}v^{\mu}h_{\lambda\mu}/2.$$

In addition to the components $h_{\lambda\mu}$ we introduce the redundant components

$$h_{\mathrm{V}\,\mathrm{V}} = h_{\mathrm{VI}\,\mathrm{VI}} = -\tfrac{1}{2} \tag*{(1.7)}$$

$$\text{the remaining } h_{\Omega\mathrm{V}} = h_{\mathrm{V}\Omega} = h_{\Omega\mathrm{VI}} = h_{\mathrm{VI}\Omega} = 0.$$

The set of all these components will be symbolized by $h_{\Omega\Lambda}$ $(= h_{\Lambda\Omega})$. The *redundant nonholonomic components* h_{AB} of $h_{\Lambda\Omega}$ are defined by

$$h_{AB} = h_{\Lambda\Omega}\underset{A}{a}^{\Lambda}\underset{B}{a}^{\Omega}. \tag*{(1.8)a}$$

In particular,

(1.8)b $$h_{55} = h_{66} = -1.$$

Therefore, every vector v^{Ω}[1] satisfies the condition

(1.9) $$v^{\Lambda} v^{\Omega} h_{\Lambda\Omega} = v^{A} v^{B} h_{AB} = 0.$$

If and only if v^{Ω} is a null vector, we have

(1.10)a $$v^{\lambda} v^{\mu} h_{\lambda\mu} = v^{a} v^{b} h_{ab} = 0$$

and this relation is equivalent to

(1.10)b $$v^{\mathrm{V}} = v^{\mathrm{VI}} = v^{5} = v^{6} = 0.$$

3) All vectors ρv^{Ω} in $T_4(O)$ have the same direction. Its ideal point in $P_3(O)$ will also be denoted by v^{Ω}. Then $v^{\mathrm{I}}, \ldots, v^{\mathrm{VI}}$ have to be looked upon as homogeneous redundant projective coordinates of the point v^{Ω} in $P_3(O)$. A similar remark also holds for the redundant nonholonomic coordinates v^{A}. The quadric H in $P_3(O)$ is defined by the ratios of the numbers $h_{\Lambda\Omega}$ as well as by the ratios of the numbers h_{AB}.[2] A point v^{Ω} on H will be termed an *isotropic* point; every other point in $P_3(O)$ will be referred to as an *anisotropic* point. Thus, an isotropic point satisfies the conditions (1.10).

The vectors $\underset{i}{a}^{\nu}$, $\overset{i}{a}_{\lambda}$ in $T_4(O)$ are known up to transformations I (6.9) (*Cf.* also I (9.5) and I (9.6)). Here we shall consider the transformations

(1.11)a $$'\underset{A}{a}^{\Omega} = \overset{-1}{C}{}_{A}{}^{B}\, \underset{B}{a}^{\Omega}$$

(1.11)b $$'\overset{B}{a}_{\Omega} = C_{A}{}^{B}\, \overset{A}{a}_{\Omega}$$

with

(1.11)c $$((C_{A}{}^{B})) \overset{\text{def}}{=} \left(\!\!\left(\begin{matrix} e^{-i\varphi} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\varphi} & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon' e^{-\psi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon' e^{\psi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}\right)\!\!\right)$$

---

[1] We do not consider $\underset{5}{a}^{\Omega}$ and $\underset{6}{a}^{\Omega}$ to be vectors.

[2] In other words $h_{\Lambda\Omega}$ (h_{AB}) and $\varrho h_{\Lambda\Omega}$ (ϱh_{AB}), $\varrho \neq 0$, define the same quadric.

$$(1.11)\text{d} \quad ((\overset{-1}{C}_A{}^B)) \overset{\text{def}}{=} \left(\left(\begin{matrix} e^{i\varphi} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-i\varphi} & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon' e^{\psi} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon' e^{-\psi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}\right)\right), \quad (\epsilon' = \pm 1).$$

Then we have from

$$(1.12) \qquad v^{\Omega} = v^A \underset{A}{a}{}^{\Omega} = {}'v^A \, {}'\underset{A}{a}{}^{\Omega}$$

the relations

$$(1.11)\text{e} \qquad {}'v^B = C_A{}^B v^A, \quad v^B = \overset{-1}{C}_A{}^B \, {}'v^A.$$

The equations (1.11)e represent transformations of nonholonomic redundant coordinates: they establish the relationship between the coordinates v^A and ${}'v^A$ involved in (1.12).

4) In subsequent considerations we shall also refer $P_3(O)$ to another set of projective homogeneous coordinates denoted by $\mathscr{X}^{\mathfrak{a}}$. The transformation (1.11) induces a projective transformation of coordinates

$$(1.13) \qquad {}'\mathscr{X}^{\mathfrak{b}} = \mathscr{C}_{\mathfrak{a}}{}^{\mathfrak{b}} \, \mathscr{X}^{\mathfrak{a}}$$

quite different from (1.11). The coefficients $\mathscr{C}_{\mathfrak{a}}{}^{\mathfrak{b}}$ of (1.13) will be found later. The space $P_3(O)$ with the coordinates $\mathscr{X}^{\mathfrak{a}}$ which are subject to the transformations (1.13) will be referred to as *the local spinor space* $\mathscr{S}_3(O)$.[1] All objects of $\mathscr{S}_3(O)$ will be termed *spinor objects* whenever necessary and denoted by script letters. Very often we shall use the abbreviation *spinor* $\mathscr{X}^{\mathfrak{a}}$ instead of spinor point $\mathscr{X}^{\mathfrak{a}}$.

The relationship between $P_3(O)$ and $\mathscr{S}_3(O)$ is based on two sets of matrices. The first set is defined by

$$(1.14) \qquad ((E_{\lambda\mathfrak{a}}{}^{\mathfrak{b}})) \overset{\text{def}}{=} \left(\left(\begin{matrix} 0 & 0 & \overset{1}{a}_{\lambda} & \overset{3}{a}_{\lambda} \\ 0 & 0 & \overset{4}{a}_{\lambda} & -\overset{2}{a}_{\lambda} \\ \overset{2}{a}_{\lambda} & \overset{3}{a}_{\lambda} & 0 & 0 \\ \overset{4}{a}_{\lambda} & -\overset{1}{a}_{\lambda} & 0 & 0 \end{matrix}\right)\right).$$

Here the superscript b (the subscript a) refers to the rows (to the columns). The second set is defined by

---

[1] This somewhat vague definition of $\mathscr{S}_3(O)$ will be amplified by a detailed discussion of the relationship $P_3(O) \leftrightarrow \mathscr{S}_3(O)$ in section 8.

$$(({}^{\epsilon}U_{\Omega}{}^{\mathrm{ab}})) = (({}^{\epsilon}U_{\Omega}{}^{[\mathrm{ab}]})) \overset{\mathrm{def}}{=} ((\overset{5}{a}_{\Omega}\overset{6}{a}_{\Omega}\overset{2}{a}_{\Omega}\overset{1}{a}_{\Omega}\overset{3}{a}_{\Omega}\overset{4}{a}_{\Omega})). \tag{1.15}$$

Here the subscript Ω refers to the rows. The superscripts refer to the columns and have to be taken in the following order:

$$12 \quad 34 \quad 23 \quad 14 \quad 31 \quad 24 \tag{1.16}$$

The symbol ϵ in ${}^{\epsilon}U_{\Omega}{}^{\mathrm{ab}}$ is the same as the one involved in (1.3) and (1.5). However, we shall use the abbreviations ${}^{+}U_{\Omega}{}^{\mathrm{ab}}$ and ${}^{-}U_{\Omega}{}^{\mathrm{ab}}$ instead of ${}^{+1}U_{\Omega}{}^{\mathrm{ab}}$ and ${}^{-1}U_{\Omega}{}^{\mathrm{ab}}$.

The transformation (1.13) will be derived from the requirement that the objects

$$\mathfrak{v}_{\mathrm{a}}{}^{\mathrm{b}} \overset{\mathrm{def}}{=} E_{\lambda\mathrm{a}}{}^{\mathrm{b}} v^{\lambda} \tag{1.17}$$

$${}^{\epsilon}\mathfrak{v}^{\mathrm{ab}} \overset{\mathrm{def}}{=} {}^{\epsilon}U_{\Omega}{}^{\mathrm{ab}} v^{\Omega} \tag{1.18}$$ [1]

are tensors with respect to (1.13) for any point v^{Ω} whatsoever.

5) In order to define the main intent of the first part of this appendix, we shall denote by $M(N)$ an arbitrary object of $P_3(O)$ (on H) and by $\mathscr{M}(\mathscr{N})$ the corresponding object of $\mathscr{S}_3(O)$. The main goal of the first part of this Appendix is the investigation of the relationships

$$M \to \mathscr{M}, \quad \mathscr{M} \to M \tag{1.19}$$

$$N \to \mathscr{N}, \quad \mathscr{N} \to N. \tag{1.20}$$

The relationship (1.19), ((1.20)) will be referred to as *representation* (*mapping*). Representation and mapping lead to Dirac's equations and also to Schrödinger's wave equations.

Sections 2–14 are devoted to the algebra of the *local* spinor space $\mathscr{S}_3(O)$. The second part, comprising the last sections 15–19, deals with analysis of the spinor space $\mathscr{S}_3$, which will be defined later.

2. Elementary results. For the convenience of the reader we rewrite equation (1.17) in a more surveyable form

$$\left(\!\!\left(\begin{matrix} \mathfrak{v}_1{}^1 & \mathfrak{v}_2{}^1 & \mathfrak{v}_3{}^1 & \mathfrak{v}_4{}^1 \\ \mathfrak{v}_1{}^2 & \mathfrak{v}_2{}^2 & \mathfrak{v}_3{}^2 & \mathfrak{v}_4{}^2 \\ \mathfrak{v}_1{}^3 & \mathfrak{v}_2{}^3 & \mathfrak{v}_3{}^3 & \mathfrak{v}_4{}^3 \\ \mathfrak{v}_1{}^4 & \mathfrak{v}_2{}^4 & \mathfrak{v}_3{}^4 & \mathfrak{v}_4{}^4 \end{matrix}\right)\!\!\right) = \left(\!\!\left(\begin{matrix} 0 & 0 & v^1 & v^3 \\ 0 & 0 & v^4 & -v^2 \\ v^2 & v^3 & 0 & 0 \\ v^4 & -v^1 & 0 & 0 \end{matrix}\right)\!\!\right). \tag{2.1}$$

---

[1] We shall very often use the abbreviations ${}^{+}\mathfrak{v}^{\mathrm{ab}}$, ${}^{-}\mathfrak{v}^{\mathrm{ab}}$ instead of ${}^{+1}\mathfrak{v}^{\mathrm{ab}}$ and ${}^{-1}\mathfrak{v}^{\mathrm{ab}}$.

This is substantially the matrix used by Cartan, E., (1).

The first Theorem to be proved concerns the products of the matrices $((E_{\lambda\mathfrak{a}}{}^{\mathfrak{b}}))$.

THEOREM (2.1). *We have*

$$E_{\lambda\mathfrak{a}}{}^{\mathfrak{b}}E_{\mu\mathfrak{b}}{}^{\mathfrak{c}} + E_{\mu\mathfrak{a}}{}^{\mathfrak{b}}E_{\lambda\mathfrak{b}}{}^{\mathfrak{c}} = h_{\lambda\mu}\delta_{\mathfrak{a}}^{\mathfrak{c}}. \tag{2.2)a}$$

The proof follows by direct calculation from

$$h_{\lambda\mu} = 2\overset{1}{a}_{(\lambda}\overset{2}{a}_{\mu)} + 2\overset{3}{a}_{(\lambda}\overset{4}{a}_{\mu)}$$ [1]

and from (1.14).

NOTE. The equation (2.2)a is often used in a simplified self-explanatory form

$$2E_{(\lambda}E_{\mu)} = h_{\lambda\mu}1$$

equivalent to

$$\sqrt{2}E_{\lambda}dx^{\lambda} = ds\,1.$$ [2]

COROLLARY (2.1). *We have*

$$2v_{\mathfrak{a}}{}^{\mathfrak{b}}v_{\mathfrak{b}}{}^{\mathfrak{c}} = \delta_{\mathfrak{a}}^{\mathfrak{c}}h_{\lambda\mu}v^{\lambda}v^{\mu} = 2\omega\delta_{\mathfrak{a}}^{\mathfrak{c}}. \tag{2.2)b}$$

The proof follows from (2.1) and (1.17) or from (2.2)a and

$$2\omega = v^{\lambda}v^{\mu}h_{\lambda\mu} = v^{a}v^{b}h_{ab} = 2(v^{1}v^{2} + v^{3}v^{4}). \tag{2.3}$$ [3]

THEOREM (2.2). *We have*

$$\mathrm{Det}((v_{\mathfrak{a}}{}^{\mathfrak{b}} - \lambda\delta_{\mathfrak{a}}^{\mathfrak{b}})) = [\omega - \lambda^{2}]^{2}. \tag{2.4}$$

The proof follows by direct computation from (2.1) and from (2.3).

In the next two Theorems we introduce a kind of generalized Kronecker delta.

THEOREM (2.3)a. *Put*

$$\delta_{A}^{\mathfrak{ab}} = \delta_{A}^{[\mathfrak{ab}]} \overset{\mathrm{def}}{=} \frac{1}{\sqrt{2}}\,{}^{\varepsilon}U_{\Omega}{}^{\mathfrak{ab}}\underset{A}{a}^{\Omega}. \tag{2.5)a}$$

---

[1] Cf. I (9.3)a.

[2] This matrix relation could be used as a basic formula for metric geometry. *Cf.*, for instance, Urbah, V., (1).

[3] *Cf.* I (8.5).

Then

(2.5)b
$$\delta_5^{12} = \delta_6^{34} = \delta_2^{23} = \delta_1^{14} = \delta_3^{31} = \delta_4^{24} =$$
$$-\delta_5^{21} = -\delta_6^{43} = -\delta_2^{32} = -\delta_1^{41} = -\delta_3^{13} = -\delta_4^{42} = \frac{1}{\sqrt{2}}$$

the remaining $\delta_A^{\mathrm{ab}} = 0$.

The proof follows at once from (1.4), (1.15) and (2.5)a.

THEOREM (2.3)b. *Put*

(2.6)a
$$\delta_{\mathrm{ab}}^A = \delta_{[\mathrm{ab}]}^A \overset{\mathrm{def}}{=} \delta_A^{\mathrm{ab}}.$$

Then

(2.6)b
$$\delta^5_{12} = \delta^6_{34} = \delta^2_{23} = \delta^1_{14} = \delta^3_{31} = \delta^4_{24} =$$
$$-\delta^5_{21} = -\delta^6_{43} = -\delta^2_{32} = -\delta^1_{41} = -\delta^3_{13} = -\delta^4_{42} = \frac{1}{\sqrt{2}},$$

the remaining $\delta^A_{\mathrm{ab}} = 0$

and

(2.7)a
$$\delta^A_{\mathrm{ab}}\delta^{\mathrm{ab}}_B = \delta^A_B,$$

(2.7)b
$$\delta^A_{\mathrm{ab}}\delta^{\mathrm{cd}}_A = \delta^{\ \mathrm{cd}}_{[\mathrm{ab}]}.$$

The proof of (2.6)b is obvious. The proof of (2.7) follows by direct computation from (2.5)b and (2.6)b.

THEOREM (2.4). *The quantities* ${}^{\epsilon}v^{\mathrm{ab}}$ *defined by* (1.18) *may be expressed as follows*

(2.8)a
$${}^{\epsilon}v^{\mathrm{ab}} = \sqrt{2}\,\delta^{\mathrm{ab}}_A v^A.$$

Moreover,

(2.8)b
$$\sqrt{2}\,v^A = \delta^A_{\mathrm{ab}}\,{}^{\epsilon}v^{\mathrm{ab}}.$$

PROOF. We obtain from (1.6)a, (1.18) and (2.5)a

$${}^{\epsilon}v^{\mathrm{ab}} = {}^{\epsilon}U_\Omega{}^{\mathrm{ab}}v^\Omega = {}^{\epsilon}U_\Omega{}^{\mathrm{ab}}v^A \underset{A}{a}{}^\Omega = \sqrt{2}\,\delta^{\mathrm{ab}}_A v^A.$$

The equation (2.8)b follows from (2.8)a and (2.7)b.

REMARK. In subsequent considerations we shall use tensors i_{abcd}, $\mathscr{I}^{\mathrm{abcd}}$ called indicators of $\mathscr{S}_3(O)$. They are skew symmetric in all four indices. The value of the component i_{abcd} as well as of the component $\mathscr{J}^{\mathrm{abcd}}$ is $+1$ (-1) when the permutation abcd is an even (an odd) permutation of *1234*. In all remaining cases it is equal to zero. This definition is invariant under the transformation (1.13): We shall see later that the components of indicators are invariant under (1.13). These indicators serve for raising and lowering the indices of quadratic skew symmetric tensors according to the rule

$$\mathfrak{t}_{\mathrm{ab}} = \tfrac{1}{2} i_{\mathrm{abcd}} \mathfrak{t}^{\mathrm{cd}} \qquad (2.9)$$
$$\mathfrak{t}^{\mathrm{ab}} = \tfrac{1}{2} \mathscr{J}^{\mathrm{abcd}} \mathfrak{t}_{\mathrm{cd}}.$$

Using this rule we easily obtain from (2.8)a

$$ {}^{\epsilon}\mathfrak{v}^{\mathrm{ab}}(v^5,\ v^6,\ v^2,\ v^1,\ v^3,\ v^4) \qquad (2.10)$$
$$ {}^{\epsilon}\mathfrak{v}_{\mathrm{ab}}(v^6,\ v^5,\ v^1,\ v^2,\ v^4,\ v^3).$$ [1] [2]

We shall use this result in

THEOREM (2.5). *${}^{\epsilon}\mathfrak{v}^{\mathrm{ab}}$ defined by* (2.8)a *is a spinor line. If v^{Ω} is an anisotropic point, then the spinor lines ${}^{+}\mathfrak{v}^{\mathrm{ab}}$ and ${}^{-}\mathfrak{v}^{\mathrm{ab}}$ are skew. If v^{Ω} is an isotropic point, then the lines ${}^{+}\mathfrak{v}^{\mathrm{ab}}$ and ${}^{-}\mathfrak{v}^{\mathrm{ab}}$ coincide:*

$$(2.11) \qquad \left.\begin{array}{l} \mathfrak{v}^{\mathrm{ab}} \overset{\mathrm{def}}{=} {}^{+}\mathfrak{v}^{\mathrm{ab}} = {}^{-}\mathfrak{v}^{\mathrm{ab}} \\ \mathfrak{v}^{\mathrm{ab}}(0,\ 0,\ v^2,\ v^1,\ v^3,\ v^4) \\ \mathfrak{v}_{\mathrm{ab}}(0,\ 0,\ v^1,\ v^2,\ v^4,\ v^3) \end{array}\right\} \ (v^{\Omega}\ \textit{isotropic})$$

PROOF. We obtain from (2.10), (1.6)b, (1.8)b, (1.9) and (2.3)

$$ \tfrac{1}{2}{}^{\epsilon}\mathfrak{v}_{\mathrm{ab}}{}^{\epsilon}\mathfrak{v}^{\mathrm{ab}} = 2v^5v^6 + h_{ab}v^a v^b \qquad (2.12)$$
$$ = h_{AB}v^A v^B = 0.$$

Hence, ${}^{\epsilon}\mathfrak{v}^{\mathrm{ab}}$ is a line. If v^{Ω} is isotropic, then we see from (2.10) and (1.10)b that (2.11) holds. In order to prove the remaining statement,

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[1] Whenever we explicitly display the components of a quadratic skew symmetric tensor $\mathfrak{t}^{\mathrm{ab}}$ of $\mathscr{S}_3(O)$ by a symbol such as

$$\mathfrak{t}^{\mathrm{ab}}(m,\ n,\ p,\ q,\ r,\ s),$$

these components have to be taken in the order indicated by (1.16), $\mathfrak{t}^{12} = m$, $\ldots$, $\mathfrak{t}^{24} = s$. A similar remark holds for

$$\mathfrak{t}_{\mathrm{ab}}(m,\ n,\ p,\ q,\ r,\ s).$$

[2] If we multiply v^{ν} or v^{a} by a factor $\varrho \neq 0$ then we put by definition $\sqrt{\varrho^2} \overset{\mathrm{def}}{=} \varrho$. Hence, if $\varrho < 0$ this definition is not in accordance with our general agreement that square roots are to be taken as positive.

we transcribe (2.10) by means of (1.6)b

$$(2.13)\qquad \begin{aligned} &{}^{+}\mathfrak{v}^{\mathrm{ab}}(-\sqrt{\omega},\ \sqrt{\omega},\ v^2,\ v^1,\ v^3,\ v^4)\\ &{}^{-}\mathfrak{v}_{\mathrm{ab}}(-\sqrt{\omega},\ \sqrt{\omega},\ v^1,\ v^2,\ v^4,\ v^3).\end{aligned}$$

Therefore,

$$(2.14)\qquad \begin{aligned} \tfrac{1}{2}\,{}^{+}\mathfrak{v}^{\mathrm{ab}}\,{}^{-}\mathfrak{v}_{\mathrm{ab}} &= 2\omega + v^a v^b h_{ab}\\ &= 2v^\lambda v^\mu h_{\lambda\mu}.\end{aligned}$$

Thus, if v^Ω is anisotropic, the last term in (2.14) is different from zero, and consequently, the spinor lines ${}^{+}\mathfrak{v}^{\mathrm{ab}}$ and ${}^{-}\mathfrak{v}^{\mathrm{ab}}$ are skew.

3. Mapping of isotropic points. In the next theorem we shall consider the projectivity

$$(3.1)\qquad {}'y^{\mathrm{b}} = \mathfrak{v}_{\mathrm{a}}{}^{\mathrm{b}}\, y^{\mathrm{a}}$$

where $\mathfrak{v}_{\mathrm{a}}{}^{\mathrm{b}}$ is given by (1.17) (or (2.1)).

THEOREM (3.1). *A necessary and sufficient condition that the projectivity* (3.1) *be a degenerate one is that the point* v^Ω *be isotropic. If this condition is satisfied, then the spinor line* (2.11) *has the following properties:*

a) *Every spinor* y^{a} *not incident with* $\mathfrak{v}^{\mathrm{ab}}$ *is carried by* (3.1) *into a spinor* ${}'y^{\mathrm{a}}$ *incident with* $\mathfrak{v}^{\mathrm{ab}}$.

b) $\mathfrak{v}^{\mathrm{ab}}$ *is the locus of spinors* y^{a} *which do not have corresponding spinors* ${}'y^{\mathrm{a}}$, *i.e., if*

$$(3.2)\text{a}\qquad \mathfrak{v}_{\mathrm{ab}}\, y^{\mathrm{b}} = 0,$$

then

$$(3.2)\text{b}\qquad \mathfrak{v}_{\mathrm{a}}{}^{\mathrm{b}}\, y^{\mathrm{a}} = 0.$$

c) *The locus of all spinor lines* $\mathfrak{v}^{\mathrm{ab}}$ *corresponding to all isotropic points* v^Ω *is a linear congruence* $\mathscr{H}$ *whose axes are given by*

$$(3.3)\text{a}\qquad {}^{1}\mathscr{Q}^{\mathrm{ab}} = a\delta_5^{\mathrm{ab}}$$

$$(3.3)\text{b}\qquad {}^{2}\mathscr{Q}^{\mathrm{ab}} = b\delta_6^{\mathrm{ab}}$$

where $a \neq 0$, $b \neq 0$ *are two arbitrary factors of proportionality.*

PROOF. We have from (3.3), (2.11)

$$(3.4)\qquad \mathfrak{v}_{\mathrm{ab}}\,{}^{1}\mathscr{Q}^{\mathrm{ab}} = \mathfrak{v}_{\mathrm{ab}}\,{}^{2}\mathscr{Q}^{\mathrm{ab}} = 0.$$

Hence, $\mathfrak{v}^{\mathfrak{ab}}$ belongs to the congruence mentioned in statement c). A generic ruling $p_{\mathfrak{ab}}$ of this congruence is obviously given by

(3.5)a $$p_{\mathfrak{ab}}(0,\ 0,\ a,\ b,\ c,\ d)$$ [1]

where a, b, c, d are four arbitrary numbers not all equal to zero which satisfy the condition

(3.5)b $$ab + cd = 0.$$

Comparing this result with

(3.6)a $$\mathfrak{v}_{\mathfrak{ab}}(0,\ 0,\ v^1,\ v^2,\ v^4,\ v^3)$$

(3.6)b $$v^1 v^2 + v^3 v^4 = 0$$

we see that every ruling $p_{\mathfrak{ab}}$ of our congruence $\mathscr{H}$ is a spinor line $\mathfrak{v}_{\mathfrak{ab}}$ and vice versa. Therefore, the locus of the spinor lines $\mathfrak{v}_{\mathfrak{ab}}$ is the congruence $\mathscr{H}$.

If we put $\lambda = 0$ in (2.4), we obtain

$$\mathrm{Det}((\mathfrak{v}_{\mathfrak{a}}{}^{\mathfrak{b}})) = \omega^2,$$

and this equation shows that (3.1) is a degenerate projectivity if and only if v^Ω is isotropic. From (2.1) we see that, for an isotropic point, the rank of the matrix $((\mathfrak{v}_{\mathfrak{a}}{}^{\mathfrak{b}}))$ is two. Therefore, the locus of spinors $y^{\mathfrak{a}}$ satisfying (3.2)b must be a spinor line. Direct computation based on (2.1), (1.10)b and (2.11) shows that

(3.7) $$\mathfrak{v}_{\mathfrak{a}}{}^{\mathfrak{b}}\mathfrak{v}^{\mathfrak{ac}} = 0.$$

Thus, the locus of spinors $y^{\mathfrak{a}}$ satisfying (3.2)b is the spinor line $\mathfrak{v}^{\mathfrak{ab}}$. Multiplying (3.1) by $\mathfrak{v}_{\mathfrak{b}}{}^{\mathfrak{c}}$ we obtain by virtue of (2.2)b for an isotropic point

$$'y^{\mathfrak{b}}\mathfrak{v}_{\mathfrak{b}}{}^{\mathfrak{c}} = \mathfrak{v}_{\mathfrak{a}}{}^{\mathfrak{b}}\mathfrak{v}_{\mathfrak{b}}{}^{\mathfrak{c}}y^{\mathfrak{a}} = 0.$$

Hence, $'y^{\mathfrak{b}}$ must be incident with $\mathfrak{v}^{\mathfrak{ab}}$.

Previous results justify the following

DEFINITION (3.1). *If v^Ω is an isotropic point, then the equations*

(3.8)a $$\mathfrak{v}^{\mathfrak{ab}} = \sqrt{2}\,\delta^{\mathfrak{ab}}_{A}v^{A}$$ [2]

(3.8)b $$\sqrt{2}\,v^{A} = \delta^{A}_{\mathfrak{ab}}\mathfrak{v}^{\mathfrak{ab}}$$

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[1] The a and b occurring in (3.5) have no relation with the a,b of (3.3).

[2] See (2.8) and (2.11).

are referred to as mappings of points of H onto rulings of $\mathscr{H}$ and vice versa.

REMARK. The four equations (3.2)b reduce to two independent equations

$$v^1 y^3 + v^3 y^4 = 0 \qquad (3.9)$$
$$v^2 y^1 + v^3 y^2 = 0$$

of the spinor line v^{ab}. Remembering that the coordinates v^1 and v^2 of *a real* isotropic point are complex conjugates, $v^2 = \bar{v}^1$, $v^1 = \bar{v}^2$, [1] we obtain from (3.9) for a *real* isotropic point v^a

$$y^4/y^3 = \bar{y}^2/\bar{y}^1 \qquad (3.10)\text{a}$$

and sometimes even

$$y^4 = \bar{y}^2 \quad y^3 = \bar{y}^1 \qquad (3.10)\text{b}$$
$$y^2 = \bar{y}^4 \quad y^1 = \bar{y}^3.$$

A spinor y^a satisfying (3.10)b will be termed a *physical* spinor. Any other spinor y^a will be referred to as an *analytical* spinor. [2]

THEOREM (3.2). *The spinor line* $\underset{i}{a}^{bc}$ *in* $\mathscr{S}_3(O)$ *mapping the isotropic point* $\underset{i}{a}^{\Omega}$ *is*

$$\underset{i}{a}^{bc} = \sqrt{2}\,\delta_i^{bc}. \qquad (3.11)$$

PROOF. One obtains (3.11) from (3.8)a and from

$$\underset{i}{a}^{A} = \underset{i}{a}^{\Omega}\overset{A}{a}_{\Omega} = \delta_i^A. \qquad (3.12)$$

NOTE. According to (3.3) and (3.11) the spinor lines ${}^1\mathscr{Q}^{ab}$, ${}^2\mathscr{Q}^{ab}$ and $\underset{i}{a}^{ab}$ are the edges of the coordinate tetrahedron in $\mathscr{S}_3(O)$.

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[1] The vectors $\overset{1}{a}_\lambda$ and $\overset{2}{a}_\lambda$ involved in (1.6)a are complex conjugates.

[2] A *physical* spinor $\mathscr{X}^a$ is represented in the two-component spinor algebra by the spinor $(\mathscr{X}^1, \mathscr{X}^2)$ and its complex-conjugate spinor $(\bar{\mathscr{X}}^1, \bar{\mathscr{X}}^2)$. An *analytical* spinor is represented by two spinors $(\mathscr{X}^1, \mathscr{X}^2)$, $(\mathscr{X}^3, \mathscr{X}^4)$ and their complex conjugates $(\bar{\mathscr{X}}^1, \bar{\mathscr{X}}^2)$, $(\bar{\mathscr{X}}^3, \bar{\mathscr{X}}^4)$.

4. Continuation. The following two Theorems will be proved simultaneously.

THEOREM (4.1)a. *At least one of the two spinors*

$$ {}^1\mathfrak{v}^{\mathrm{a}} \overset{\text{def}}{=} \mathfrak{v}^{\mathrm{a}4} \quad \textit{or} \quad {}^1\mathfrak{v}^{\mathrm{a}} \overset{\text{def}}{=} \mathfrak{v}^{\mathrm{a}3} \tag{4.1)a} $$

exists on the spinor line $\mathfrak{v}^{\mathrm{ab}}$ *given by* (2.11). *If both exist, they are identical. The spinor* ${}^1\mathfrak{v}^{\mathrm{a}}$ *is the focal point of* $\mathfrak{v}^{\mathrm{ab}}$ *on* ${}^1\mathscr{Q}^{\mathrm{ab}}$.

THEOREM (4.1)b. *At least one of the two spinors*

$$ {}^2\mathfrak{v}^{\mathrm{a}} \overset{\text{def}}{=} \mathfrak{v}^{\mathrm{a}1} \quad \textit{or} \quad {}^2\mathfrak{v}^{\mathrm{a}} \overset{\text{def}}{=} \mathfrak{v}^{\mathrm{a}2} \tag{4.1)b} $$

exists on $\mathfrak{v}^{\mathrm{ab}}$. *If both exist, they are identical. The spinor* ${}^2\mathfrak{v}^{\mathrm{a}}$ *is the focal point of* $\mathfrak{v}^{\mathrm{ab}}$ *on* ${}^2\mathscr{Q}^{\mathrm{ab}}$.

PROOF. The axis ${}^1\mathscr{Q}^{\mathrm{ab}}$ of $\mathscr{H}$ is the intersection line of the spinor planes

$$ \mathscr{X}^{\mathrm{f}} = 0 \qquad \mathrm{f} = 3,\ 4. \tag{4.2} $$

The ruling $\mathfrak{v}^{\mathrm{ab}}$ of $\mathscr{H}$ intersects ${}^1\mathscr{Q}^{\mathrm{ab}}$ in one spinor point and, therefore, can not be incident with both spinor planes (4.2). If $\mathfrak{v}^{\mathrm{ab}}$ is not incident with the spinor plane (4.2) for $\mathrm{f} = 3$ or $\mathrm{f} = 4$, then its intersection with this plane is the spinor $\mathfrak{v}^{\mathrm{af}}$ ($\mathrm{f} = 3$ or $\mathrm{f} = 4$), and this spinor is incident with ${}^1\mathscr{Q}^{\mathrm{ab}}$. Theorem (4.1)a follows easily from these facts. Theorem (4.1)b can be proved in a similar way.

NOTE. We see from (2.11) that the spinors (4.1) are

$$ \text{either } {}^1\mathfrak{v}^{\mathrm{a}}(v^1, v^4, 0, 0) \text{ or } {}^1\mathfrak{v}^{\mathrm{a}}(-v^3, v^2, 0, 0) \tag{4.3)a} $$

$$ \text{either } {}^2\mathfrak{v}^{\mathrm{a}}(0, 0, v^3, -v^1) \text{ or } {}^2\mathfrak{v}^{\mathrm{a}}(0, 0, v^2, v^4). \tag{4.3)b} $$

The point v^{Ω} being an isotropic one, we see from (4.3) and (2.3) that if both spinors ${}^1\mathfrak{v}^{\mathrm{a}}$ (${}^2\mathfrak{v}^{\mathrm{a}}$) exist, they are identical.

COROLLARY (4.1). *The focal spinors* ${}^1\underset{i}{a}^{\mathrm{b}}$, ${}^2\underset{i}{a}^{\mathrm{b}}$ *of the spinor lines* $\underset{i}{a}^{\mathrm{ab}}$ *given by* (3.11) *are*

$$ \begin{aligned} {}^1\underset{1}{a}^{\mathrm{b}}(1, 0, 0, 0) &= {}^1\underset{3}{a}^{\mathrm{b}} \\ {}^1\underset{2}{a}^{\mathrm{b}}(0, 1, 0, 0) &= {}^1\underset{4}{a}^{\mathrm{b}} \\ {}^2\underset{3}{a}^{\mathrm{b}}(0, 0, 1, 0) &= {}^2\underset{2}{a}^{\mathrm{b}} \\ {}^2\underset{4}{a}^{\mathrm{b}}(0, 0, 0, 1) &= {}^2\underset{1}{a}^{\mathrm{b}}. \end{aligned} \tag{4.4} $$

PROOF. Substituting into (4.3) $v^j = \underset{i}{a}^j$ and using (3.12), one obtains (4.4).

The following two Theorems will be proved simultaneously.

THEOREM (4.2). *Let* y^Ω *be a generic point of a ruling* R *of* H *which belongs to the same regulus as* $\underset{1}{a}^{[\nu}\underset{3}{a}^{\lambda]}$ *(as* $\underset{1}{a}^{[\nu}\underset{4}{a}^{\lambda]}$*). The set of isotropic points* y^Ω *on* R *is mapped onto a pencil* $\mathcal{R}$ *of spinor lines* y^{ab} *of* $\mathcal{H}$*. The vertex of* $\mathcal{R}$ *is the common focal spinor* ${}^1y^{\mathrm{a}}$ *(*${}^2y^{\mathrm{a}}$*) on* ${}^1\mathcal{Q}^{\mathrm{ab}}$ *(*${}^2\mathcal{Q}^{\mathrm{ab}}$*) of all spinor lines* y^{ab}*. The plane of* $\mathcal{R}$ *is defined by* ${}^1y^{\mathrm{a}}$ *and* ${}^2\mathcal{Q}^{\mathrm{ab}}$ *(by* ${}^2y^{\mathrm{a}}$ *and* ${}^1\mathcal{Q}^{\mathrm{ab}}$*).*

THEOREM (4.3). *The ruling* R *of the previous Theorem is mapped into* ${}^1y^{\mathrm{a}}$ *(*${}^2y^{\mathrm{a}}$*) on* ${}^1\mathcal{Q}^{\mathrm{ab}}$ *(on* ${}^2\mathcal{Q}^{\mathrm{ab}}$*) and into its focal plane.*

PROOF. Let p^Ω and q^Ω be two arbitrary points on a ruling of R. These two points are mapped on the spinor lines

$$p^{\mathrm{ab}}(0,\ 0,\ p^2,\ p^1,\ p^3,\ p^4) \tag{4.5}$$

$$p_{\mathrm{ab}}(0,\ 0,\ p^1,\ p^2,\ p^4,\ p^3)$$

$$q^{\mathrm{ab}}(0,\ 0,\ q^2,\ q^1,\ q^3,\ q^4) \tag{4.6}$$

$$q_{\mathrm{ab}}(0,\ 0,\ q^1,\ q^2,\ q^4,\ q^3).$$

Put

$$y^\Omega = \lambda p^\Omega + \mu q^\Omega. \tag{4.7}$$

The corresponding line is

$$y^{\mathrm{ab}}(0,\ 0,\ \lambda p^2 + \mu q^2,\ \lambda p^1 + \mu q^1,\ \lambda p^3 + \mu q^3,\ \lambda p^4 + \mu q^4) \tag{4.8)a}$$

$$y_{\mathrm{ab}}(0,\ 0,\ \lambda p^1 + \mu q^1,\ \lambda p^2 + \mu q^2,\ \lambda p^4 + \mu q^4,\ \lambda p^3 + \mu q^3)$$

or

$$y^{\mathrm{ab}} = \lambda p^{\mathrm{ab}} + \mu q^{\mathrm{ab}}. \tag{4.8)b}$$

From

$$p^a p^b h_{ab} = q^a q^b h_{ab} = p^a q^b h_{ab} = 0 \tag{4.9}$$

we see that (4.8) represents a pencil $\mathcal{R}$ of lines [1] which are rulings

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[1] y^{ab} given by (4.8)b is clearly a line and so is

$$'y^{\mathrm{ab}} = {}'\lambda p^{\mathrm{ab}} + {}'\mu q^{\mathrm{ab}}.$$

From (4.9), (4.5) and (4.6) we obtain

$$\tfrac{1}{2} y_{\mathrm{ab}}\,'y^{\mathrm{ab}} = \tfrac{1}{2}('\mu\lambda + {}'\lambda\mu)\, p_{\mathrm{ab}} q^{\mathrm{ab}} = \tfrac{1}{2}('\mu\lambda + {}'\lambda\mu) p^a q^b h_{ab} = 0.$$

of $\mathscr{H}$. Hence, the vertex of $\mathscr{R}$ is either on ${}^1\mathscr{Q}^{ab}$ or on ${}^2\mathscr{Q}^{ab}$. In order to decide which one of these axes of $\mathscr{H}$ is involved, consider the ruling $\underset{1}{a}^{[\nu}\underset{3}{a}^{\lambda]}$ of H. The spinor lines $\underset{1}{a}^{bc}$ and $\underset{3}{a}^{ac}$ mapping the points $\underset{1}{a}^{\Omega}$ and $\underset{3}{a}^{\Omega}$ respectively have the common focal spinor ${}^1\underset{1}{a}^{b} = {}^1\underset{3}{a}^{b}$ on ${}^1\mathscr{Q}^{ab}$ (*Cf.* (4.4)). Therefore, if R belongs to the same regulus as $\underset{1}{a}^{[\nu}\underset{3}{a}^{\lambda]}$, the common focal spinor of the lines (4.8)b must, by reason of continuity, be incident with ${}^1\mathscr{Q}^{ab}$. The plane of $\mathscr{R}$ is defined by this common focal spinor and by the second axis ${}^2\mathscr{Q}^{ab}$ of $\mathscr{H}$, because all lines of $\mathscr{R}$ are rulings of $\mathscr{H}$ and meet ${}^2\mathscr{Q}^{ab}$. The remaining statements of Theorem (4.2) can be proved in a similar way. According to this Theorem the common support R of points (4.7) is mapped into the common support of the pencil $\mathscr{R}$ of the lines (4.8), *i.e.*, into its vertex and its plane.

NOTE I. Given a spinor ${}^1y^{a}$ (${}^2y^{a}$) on ${}^1\mathscr{Q}^{ab}$ (on ${}^2\mathscr{Q}^{ab}$) its focal plane is uniquely determined by this spinor and the axis ${}^2\mathscr{Q}^{ab}$ (${}^1\mathscr{Q}^{ab}$). Therefore, very often we shall abbreviate by saying that an isotropic line (*i.e.*, a ruling of H) is mapped into a focal spinor, neglecting the corresponding focal plane of this focal spinor.

NOTE II. From previous results we see that

a) the quadric H considered as a two-parameter locus of isotropic points is mapped onto the congruence $\mathscr{H}$,

b) the regulus ${}^1Q$ (${}^2Q$) of isotropic lines, considered as a one-parameter set of lines containing $\underset{1}{a}^{[\nu}\underset{3}{a}^{\lambda]}$ ($\underset{1}{a}^{[\nu}\underset{4}{a}^{\lambda]}$) is mapped onto the one-parameter set of spinors of the axis ${}^1\mathscr{Q}^{ab}$ (${}^2\mathscr{Q}^{ab}$).

5. Spinor coordinate transformations. In this section we shall investigate the spinor coordinate transformation (1.13), *i.e.*,

$$(5.1)\qquad {}'\mathscr{X}^{b} = \mathscr{C}_{a}{}^{b}\,\mathscr{X}^{a}, \qquad \mathscr{X}^{b} = \overset{-1}{\mathscr{C}_{a}{}^{b}}\,{}'\mathscr{X}^{a}$$

induced in $\mathscr{S}_3(O)$ by the transformations (1.11) of nonholonomic coordinates in $P_3(O)$.

A tensor with respect to the transformation (1.11) ((5.1)) will be called for short a C-tensor (a $\mathscr{C}$-tensor). By a $(C, \mathscr{C})$-tensor we shall mean a set of components which are transformed as components of a product of a C-tensor and a $\mathscr{C}$-tensor. We shall use this conception in the next

THEOREM (5.1). *If we regard* δ_A^{ab} *as a* $(C, \mathscr{C})$*-tensor whose components are the same in all coordinate systems*

$$'\delta_A^{\mathrm{ab}} = \delta_A^{\mathrm{ab}}, \tag{5.2}$$

then

$$C_A{}^B = \delta_{\mathrm{ab}}^B \delta_A^{\mathrm{cd}} \mathscr{C}_{\mathrm{c}}{}^{\mathrm{a}} \mathscr{C}_{\mathrm{d}}{}^{\mathrm{b}} \tag{5.3)a}$$

$$\mathscr{C}_{[\mathrm{c}}{}^{\mathrm{a}} \mathscr{C}_{\mathrm{d}]}{}^{\mathrm{b}} = C_A{}^B \delta_B^{\mathrm{ab}} \delta_{\mathrm{cd}}^A. \tag{5.3)b}$$

PROOF. According to our assumption δ_A^{ab} is a $(C, \mathscr{C})$-tensor. Therefore, we obtain by virtue of (5.2)

$$'\delta_A^{\mathrm{ab}} = \delta_A^{\mathrm{ab}} = \overset{-1}{C}_A{}^B \mathscr{C}_{\mathrm{c}}{}^{\mathrm{a}} \mathscr{C}_{\mathrm{d}}{}^{\mathrm{b}} \delta_B^{\mathrm{cd}}. \tag{5.4}$$

This equation is equivalent to (5.3)a. This latter equation together with (2.7) leads to (5.3)b.

NOTE I. An easy inspection based on (5.3) shows that

a) δ_{ab}^A is a $(C, \mathscr{C})$-tensor with the property

$$'\delta_{\mathrm{ab}}^A = \delta_{\mathrm{ab}}^A. \tag{5.5)a}$$

b) if we require that the equations (2.8) hold in all coordinate systems, we obtain according to (5.3)a and (2.7)

$$\begin{aligned} '\epsilon_{\mathfrak{v}}{}^{\mathrm{ab}} &= \sqrt{2}\,'\delta_A^{\mathrm{ab}}\,'v^A = \sqrt{2}\,\delta_A^{\mathrm{ab}} C_B{}^A v^B \\ &= \sqrt{2}\,\delta_A^{\mathrm{ab}} \delta_B^{\mathrm{cd}} \delta_{\mathrm{cf}}^A \mathscr{C}_{\mathrm{c}}{}^{\mathrm{e}} \mathscr{C}_{\mathrm{d}}{}^{\mathrm{f}} \left(\frac{1}{\sqrt{2}}\, \delta_{\mathrm{ij}}^B\ \epsilon_{\mathfrak{v}}{}^{\mathrm{ij}} \right) \\ &= \delta_{[\mathrm{ef}]}^{\mathrm{ab}|} \delta_{[\mathrm{i\,j}]}^{\mathrm{cd}} \mathscr{C}_{\mathrm{c}}{}^{\mathrm{e}} \mathscr{C}_{\mathrm{d}}{}^{\mathrm{f}}\ \epsilon_{\mathfrak{v}}{}^{\mathrm{ij}} \\ &= \mathscr{C}_{\mathrm{i}}{}^{\mathrm{a}} \mathscr{C}_{\mathrm{j}}{}^{\mathrm{b}}\ \epsilon_{\mathfrak{v}}{}^{\mathrm{ij}}. \end{aligned} \tag{5.5)b}$$

Hence, $\epsilon_{\mathfrak{v}}{}^{\mathrm{ab}}$ is a $\mathscr{C}$-tensor.

NOTE II. Although the transformation (1.11) ((5.1)) represents a coordinate transformation we may regard it as a transformation carrying a C-tensor ($\mathscr{C}$-tensor) into another tensor. This point of view does not restrict the generality and enables us to speak of objects preserved by this transformation. Thus, for instance, we say that the ruling $\underset{1}{a}{}^{[\nu}\underset{3}{a}{}^{\lambda]}$ is preserved by (1.11).[1] Then, of course, we expect that its map ${}^1\underset{1}{a}{}^b = {}^1\underset{3}{a}{}^b$ (see (4.4)) is preserved by (5.1) whenever (5.3) holds. However, this is not obvious and has to be proved.

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[1] instead of saying that the ratios of the coordinates of $\underset{1}{a}{}^{[\nu}\underset{3}{a}{}^{\lambda]}$ are preserved by (1.11).

THEOREM (5.2). *The focal spinors* (4.4) *are preserved by the transformations* (5.1) *for which* (5.3) *holds.*

PROOF. The transformations (1.11) preserve the quadric H. Thus, an isotropic point is carried by (1.11) into an isotropic point. Therefore, if v^A is an isotropic point, we have by virtue of the statement b) of the preceding Note I

$$\text{(5.6)a} \qquad \mathfrak{v}^{ab} = \sqrt{2}\,\delta_A^{ab} v^A,$$

and also

$$\text{(5.6)b} \qquad {}'\mathfrak{v}^{ab} = \sqrt{2}\,\delta_A^{ab}\,{}'v^A \quad (= \sqrt{2}\,\delta_A^{ab} C_B{}^A v^B).$$

If the spinor ${}^1\mathfrak{v}^a \overset{\text{def}}{=} {}^1\mathfrak{v}^{a4}$ exists, it is, by virtue of Theorem (4.1)a, one of the focal spinors of $\mathfrak{v}^{ab}$. According to (5.6)b the spinor ${}'^1\mathfrak{v}^a \overset{\text{def}}{=} {}'\mathfrak{v}^{a4}$ is one of the focal spinors of the spinor line ${}'\mathfrak{v}^{ab}$. From (5.6)b we obtain by virtue of (2.5) and (1.11)c

$${}'^1\mathfrak{v}^1 : {}'^1\mathfrak{v}^2 : {}'^1\mathfrak{v}^3 : {}'^1\mathfrak{v}^4 = C_1{}^1 v^1 : C_4{}^4 v^4 : 0 : 0.$$

Hence, the focal spinor ${}'^1\mathfrak{v}^a$ is again incident with the axis ${}^1\mathscr{Q}^{ab}$, and therefore, ${}^1\mathscr{Q}^{ab}$ must be preserved by (5.1). On the other hand, we see from (3.11) and (5.2) that the spinor line $\underset{1}{a}{}^{ab}$ is preserved. Therefore, its focal spinor ${}^1\underset{1}{a}{}^b$ on ${}^1\mathscr{Q}^{ab}$ must also be preserved by (5.1). In a similar way we may prove that all four focal spinors (4.4) are preserved by (5.1).

We shall use this result for the proof of

THEOREM (5.3). *If the coefficients* $C_A{}^B$ *are given by* (1.11)c, *then*

$$\text{(5.7)a} \qquad \mathscr{C}_a{}^b \overset{\text{def}}{=} \left(\!\!\left(\begin{matrix} \epsilon_1 e^{-\Phi} & 0 & 0 & 0 \\ 0 & \epsilon_1 e^{\Phi} & 0 & 0 \\ 0 & 0 & \epsilon_2 e^{-\bar{\Phi}} & 0 \\ 0 & 0 & 0 & \epsilon_2 e^{\bar{\Phi}} \end{matrix}\right)\!\!\right)$$

$$\text{(5.7)b} \qquad \overset{-1}{\mathscr{C}}{}_a{}^b \overset{\text{def}}{=} \left(\!\!\left(\begin{matrix} \epsilon_1 e^{\Phi} & 0 & 0 & 0 \\ 0 & \epsilon_1 e^{-\Phi} & 0 & 0 \\ 0 & 0 & \epsilon_2 e^{\bar{\Phi}} & 0 \\ 0 & 0 & 0 & \epsilon_2 e^{-\bar{\Phi}} \end{matrix}\right)\!\!\right)$$

where

$$(5.7)\text{c} \quad 2\Phi \overset{\text{def}}{=} \psi + i\left[\frac{\pi}{2}(1-\epsilon')+\varphi\right], \quad 2\bar{\Phi} = \psi - i\left[\frac{\pi}{2}(1-\epsilon')+\varphi\right]$$

$$(5.7)\text{d} \qquad \epsilon_1 = \pm 1, \qquad \epsilon_2 = \pm 1.$$

PROOF. We see from Corollary (4.1) and Theorem (5.2) that

$$(5.8) \qquad \mathscr{C}_{\text{a}}{}^{\text{b}} = \left(\!\!\begin{pmatrix} \mathscr{C}_1{}^1 & 0 & 0 & 0 \\ 0 & \mathscr{C}_2{}^2 & 0 & 0 \\ 0 & 0 & \mathscr{C}_3{}^3 & 0 \\ 0 & 0 & 0 & \mathscr{C}_4{}^4 \end{pmatrix}\!\!\right).$$

Substituting these values into (5.3)a, we obtain by virtue of (1.11)c, (2.5)b and (2.6)a for $A = B = 6, 5, 4, 2, 3, 1$

$$(5.9) \qquad \begin{aligned} 1 &= \mathscr{C}_3{}^3\mathscr{C}_4{}^4 \\ 1 &= \mathscr{C}_1{}^1\mathscr{C}_2{}^2 \\ \epsilon' e^{\psi} &= \mathscr{C}_2{}^2\mathscr{C}_4{}^4 = e^{\psi + \frac{i\pi}{2}(\epsilon'-1)} \\ e^{i\varphi} &= \mathscr{C}_2{}^2\mathscr{C}_3{}^3 \\ \epsilon' e^{-\psi} &= \mathscr{C}_1{}^1\mathscr{C}_3{}^3 = e^{-[\psi + \frac{i\pi}{2}(\epsilon'-1)]} \\ e^{-i\varphi} &= \mathscr{C}_1{}^1\mathscr{C}_4{}^4. \end{aligned}$$

The first two equations (5.9) yield

$$(5.10) \qquad \begin{aligned} \mathscr{C}_1{}^1 &= \epsilon_1 e^{-\Phi} & \mathscr{C}_2{}^2 &= \epsilon_1 e^{\Phi} \\ \mathscr{C}_3{}^3 &= \epsilon_2 e^{-\Psi} & \mathscr{C}_4{}^4 &= \epsilon_2 e^{\Psi} \end{aligned}$$

where ϵ_1, ϵ_2 are defined by (5.7)d and Φ, Ψ have to be found. Substituting these values into the remaining equations (5.9), we see that the third and fourth reduce to

$$(5.11)\text{a} \qquad \begin{aligned} e^{\psi + \frac{i\pi}{2}(\epsilon'-1)} &= \epsilon_1\epsilon_2 e^{\Phi+\Psi} \\ e^{i\varphi} &= \epsilon_1\epsilon_2 e^{\Phi-\Psi}, \end{aligned}$$

while the remaining two equations (5.9) are inverse to (5.11)a. The equations (5.11)a are equivalent to

$$(5.11)\text{b} \qquad \begin{aligned} e^{\psi + i\left[\frac{\pi}{2}(\epsilon'-1)+\varphi\right]} &= e^{2\Phi} \\ e^{\psi - i\left[\frac{\pi}{2}(\epsilon'-1)+\varphi\right]} &= e^{2\Psi} \end{aligned}$$ [1]

---

[1] We are using here the identity $e^{i\pi} = e^{-i\pi}$.

and yield (5.7)c. Thus, we easily obtain (5.7)a,b from (5.7)c, (5.8) and (5.10)

DEFINITION (5.1). *The relationship* (5.3) *will be referred to as the representation of a coordinate transformation in* $P_3(O)$ *by a spinor coordinate transformation in* $\mathscr{S}_3(O)$ *and vice versa.*

NOTE I. The equations (5.7) show that the transformation (1.11) is represented by two transformations (5.1), one for $\epsilon_1\epsilon_2 = 1$ and one for $\epsilon_1\epsilon_2 = -1$. Thus, for instance, the identity in $P_3(O)$

$$\varphi = \psi = \epsilon' - 1 = 0 \tag{5.12}$$

is represented in $\mathscr{S}_3(O)$ by the identity

$$'\mathscr{X}^{\mathrm{a}} = \mathscr{X}^{\mathrm{a}}$$

and by the biaxial involution:

$$'\mathscr{X}^1 : {}'\mathscr{X}^2 : {}'\mathscr{X}^3 : {}'\mathscr{X}^4 = \mathscr{X}^1 : \mathscr{X}^2 : -\mathscr{X}^3 : -\mathscr{X}^4 \tag{5.13}$$

with the axes ${}^1\mathscr{Q}^{\mathrm{ab}}$ and ${}^2\mathscr{Q}^{\mathrm{ab}}$.

NOTE II. The equations (3.10)b are not affected by (5.1) *if and only if*

$$\epsilon_1\epsilon_2 = 1. \tag{5.14}$$

Therefore, whenever we deal explicitly with *physical* spinors we tacitly assume the validity of (5.14).

NOTE III. The transformation (5.7) is unimodular. Therefore, the components of the indicators are invariant under spinor coordinate transformations.

6. Representation of an anisotropic point.

THEOREM (6.1). *The components* $v_{\mathrm{a}}{}^{\mathrm{b}}$ *defined by* (1.17) *define a* $\mathscr{C}$*-tensor*:

$$'v_{\mathrm{a}}{}^{\mathrm{b}} = \overset{-1}{\mathscr{C}}{}_{\mathrm{a}}{}^{\mathrm{c}}\mathscr{C}_{\mathrm{d}}{}^{\mathrm{b}}v_{\mathrm{c}}{}^{\mathrm{d}}. \tag{6.1}$$

PROOF. We have from (1.14) and (1.11)a

$$'E_{\lambda 3}{}^{1} = E_{\lambda 3}{}^{1}C_1{}^1.$$

Using (5.3)a and (5.7) we may rewrite this equation as

$$'E_{\lambda 3}{}^{1} = \overset{-1}{\mathscr{C}}{}_3{}^3\mathscr{C}_1{}^1E_{\lambda 3}{}^1 = \overset{-1}{\mathscr{C}}{}_3{}^{\mathrm{c}}\mathscr{C}_{\mathrm{d}}{}^1E_{\lambda\mathrm{c}}{}^{\mathrm{d}}.$$

In a similar way we obtain

$$(6.2)\qquad 'E_{\lambda a}{}^{b} = \overset{-1}{\mathscr{C}}_{a}{}^{c}\mathscr{C}_{d}{}^{b}E_{\lambda c}{}^{d}.$$

The proof follows easily from (6.2) and (1.17).

THEOREM (6.2). *Let* v^{Ω} *be an anisotropic point. Then the* $\mathscr{C}$*-tensor* $v_{a}{}^{b}$ *defined by* (1.17) *gives rise to a projectivity*

$$(6.3)\qquad '\mathscr{X}^{b} = v_{a}{}^{b}\mathscr{X}^{a}$$

which is a biaxial involution whose skew axes are the spinor lines $^{\epsilon}v^{ab}$ *given by* (2.8)a.

PROOF. From (2.2)b we have

$$(6.4)\qquad v_{a}{}^{b}v_{b}{}^{c} = \omega\delta_{a}^{c}$$

$$(\omega \overset{\text{def}}{=} h_{\lambda\mu}v^{\lambda}v^{\mu}/2).$$

Thus, if we apply the projectivity (6.3) twice to $\mathscr{X}^{a}$ we return to $\mathscr{X}^{a}$; therefore, the projectivity (6.3) is an involution. Denote by $^{\epsilon}p^{a}$ the spinor $^{\epsilon}v^{a4}$ of the line $^{\epsilon}v^{ab}$ defined by (2.8)a,

$$(6.5)\qquad ^{\epsilon}p^{a}(v^{1}, v^{4}, v^{6}, 0).$$

Then we obtain from (6.3) and (2.1)

$$(6.6)\qquad '^{\epsilon}p^{a} = v^{6}\,{}^{\epsilon}p^{a}.$$

Hence, $^{\epsilon}p^{a}$ is preserved by (6.3). The same holds for the spinors $^{\epsilon}v^{a3}$, $^{\epsilon}v^{a2}$, $^{\epsilon}v^{a1}$. Therefore, the line $^{\epsilon}v^{ab}$ is preserved pointwise by (6.3). According to Theorem (2.5) the lines $^{+}v^{ab}$ and $^{-}v^{ab}$ are skew. These facts show that $^{\epsilon}v^{ab}$ are axes of the biaxial involution (6.3).

DEFINITION (6.1). *The relations* (2.8)a,b *will be referred to as the representation of an anisotropic point* v^{Ω} *in* $P_3(O)$ *by the spinor lines* $^{\epsilon}v^{ab}$ *in* $\mathscr{S}_3(O)$ *and vice versa.*

7. Continuation. In subsequent considerations we shall use two linear complexes: the complex $\mathfrak{k}$ whose components are

$$(7.1)\qquad \mathfrak{k}_{ab}(1, 1, 0, 0, 0, 0)$$

$$\mathfrak{k}^{ab}(1, 1, 0, 0, 0, 0)$$

and the complex $\mathscr{K}$ whose components are

$$\mathscr{K}_{\mathrm{ab}}(1,\,-1,\,0,\,0,\,0,\,0) \tag{7.2}$$

$$\mathscr{K}^{\mathrm{ab}}(-1,\,1,\,0,\,0,\,0,\,0).$$

Both complexes are general

$$k_{\mathrm{ab}}k^{\mathrm{ab}} \neq 0, \quad \mathscr{K}_{\mathrm{ab}}\mathscr{K}^{\mathrm{ab}} \neq 0. \tag{7.3}$$

Moreover, they are projectively orthogonal

$$k_{\mathrm{ab}}\mathscr{K}^{\mathrm{ab}} = k^{\mathrm{ab}}\mathscr{K}_{\mathrm{ab}} = 0. \tag{7.4}$$

The following two Theorems will be proved simultaneously.

THEOREM (7.1). *The locus of all spinor lines* ${}^{\epsilon}v^{\mathrm{ab}}$ *representing all points* v^{Ω} *of* $P_3(O)$ *is the linear complex* k_{ab}.

THEOREM (7.2). *The lines* ${}^{\epsilon}v^{\mathrm{ab}}$ *are conjugate polars with respect to the complex* $\mathscr{K}_{\mathrm{ab}}$.

PROOF. We obtain from (1.6)b, (2.10) and (7.1)

$$k_{\mathrm{ab}}\,{}^{\epsilon}v^{\mathrm{ab}} = 0. \tag{7.5}$$

Thus, every spinor line ${}^{\epsilon}v^{\mathrm{ab}}$ is a ruling of the complex k_{ab}. Every ruling p^{ab} of the complex k_{ab} has the coordinates

$$p^{\mathrm{ab}}(a,\,b,\,c,\,d,\,e,\,f) \tag{7.6a}$$

where

$$ab + cd + ef = 0, \quad a + b = 0. \tag{7.6b}$$

Comparing (7.6) with (2.10) and (1.6)b and remembering (1.9), we see that every ruling of k_{ab} may be thought of as one of the lines ${}^{\epsilon}v^{\mathrm{ab}}$. According to Theorem (2.5) these lines coincide if v^{Ω} is an isotropic point.

In order to prove Theorem (7.2) we remind the reader that polars p^{ab} and q^{ab}, with respect to a complex c_{ab}, are related by

$$p^{\mathrm{ab}} = c^{\mathrm{ab}}c_{\mathrm{cd}}q^{\mathrm{cd}} - \tfrac{1}{2}q^{\mathrm{ab}}c_{\mathrm{cd}}c^{\mathrm{cd}}. \tag{7.7}$$

Put $c_{\mathrm{ab}} = \mathscr{K}_{\mathrm{ab}}$ and $q^{\mathrm{ab}} = {}^{\epsilon}v^{\mathrm{ab}}$ into (7.7). Then we obtain according to (1.6)b, (7.2) and (2.10)

$$p^{\mathrm{ab}} = -\,4\epsilon\sqrt{\omega}\mathscr{K}^{\mathrm{ab}} + 2\,{}^{\epsilon}v^{\mathrm{ab}}$$

or

$$\tfrac{1}{2}p^{\mathrm{ab}}(\epsilon\sqrt{\omega},\ -\ \epsilon\sqrt{\omega},\ v^2,\ v^1,\ v^3,\ v^4). \tag{7.8}$$

Therefore, if $q^{\mathrm{ab}} = {}^{+}\mathfrak{v}^{\mathrm{ab}}$ $(q^{\mathrm{ab}} = {}^{-}\mathfrak{v}^{\mathrm{ab}})$, then by (2.13)

$$p^{\mathrm{ab}} = {}^{-}\mathfrak{v}^{\mathrm{ab}}\ \ (p^{\mathrm{ab}} = {}^{+}\mathfrak{v}^{\mathrm{ab}}).$$

NOTE I. The congruence $\mathscr{H}$ is an intersection of the complexes (7.1) and (7.2). An isotropic point v^{Ω} is mapped on a ruling (2.11) of $\mathscr{H}$. This is in accordance with the known fact that conjugate polars with respect to a complex coincide if and only if one of them is a ruling of this complex.

NOTE II. The spinor polar plane $\mathscr{X}_{\mathrm{a}}$ of a spinor $\mathscr{X}^{\mathrm{a}}$ with respect to the complex k is given by

$$\mathscr{X}_{\mathrm{a}} = k_{\mathrm{ab}}\mathscr{X}^{\mathrm{b}} \tag{7.9}$$

or, according to (7.1),

$$\mathscr{X}_1 = \mathscr{X}^2,\ \mathscr{X}_2 = -\ \mathscr{X}^1,\ \mathscr{X}_3 = \mathscr{X}^4,\ \mathscr{X}_4 = -\ \mathscr{X}^3. \tag{7.10}$$

If, in particular, $\mathscr{X}^{\mathrm{a}}$ is a physical spinor, *i.e.*, if (3.10)b holds, then

$$\mathscr{X}_3 = \bar{\mathscr{X}}_1,\quad \mathscr{X}_4 = \bar{\mathscr{X}}_2.$$

In the quantum theory the numbers $\mathscr{X}_1$, $\mathscr{X}_2$, $\bar{\mathscr{X}}_1$, $\bar{\mathscr{X}}_2$ are termed covariant components of the (physical) spinor $\mathscr{X}^{\mathrm{a}}$.

8. $P_3(O)$ **and** $\mathscr{S}_3(O)$. In the first chapter of this book we referred the space-time to the nonholonomic frame $\underset{i}{a^{\nu}}$, $\overset{j}{a}_{\lambda}$ of basic vectors. In the case of the first class this frame is defined up to transformations (1.11). It is useful for our purposes to refer the space-time to another nonholonomic frame

$$\underset{\mathbf{x}}{a^{\nu}} \overset{\mathrm{def}}{=} \underset{x}{\rho}\,\underset{x}{a^{\nu}},\quad \overset{\mathbf{x}}{a}_{\mu} \overset{\mathrm{def}}{=} \underset{x}{\rho}^{-1}\overset{x}{a}_{\mu} \tag{8.1}$$

where the scalars $\underset{x}{\rho}$ are defined by

$$\underset{1}{\rho}\,\underset{2}{\rho}\,\alpha = \underset{3}{\rho}\,\underset{4}{\rho}\,\beta = 1$$

and α, β are defined by I (6.2). This frame satisfies the conditions

$$\underset{\mathbf{i}}{a^{\nu}}\overset{\mathbf{i}}{a}_{\lambda} = \delta^{\nu}_{\lambda},\quad \underset{\mathbf{i}}{a^{\nu}}\overset{\mathbf{j}}{a}_{\nu} = \delta^{\mathbf{j}}_{\mathbf{i}}, \tag{8.2}$$

and it is known up to transformations

$$(8.3) \qquad {}'\underset{\mathrm{x}}{a}{}^{\nu} = \overset{-1}{C}{}_x{}^x\, \underset{\mathrm{x}}{a}{}^{\nu}, \quad {}'\overset{\mathrm{x}}{a}{}_{\lambda} = C_x{}^x\, \overset{\mathrm{x}}{a}{}_{\lambda}$$

where the C's are to be taken from the matrices (1.11)c,d. The non-holonomic components with respect to the frame $\underset{\mathrm{i}}{a}{}^{\nu}$, $\overset{\mathrm{j}}{a}{}_{\lambda}$ of an arbitrary tensor $T^{\nu\cdots}_{\lambda\cdots}$ will be denoted by

$$(8.4) \qquad T^{\mathrm{i}\cdots}_{\mathrm{j}\cdots} = \overset{\mathrm{i}}{a}{}_{\nu} \underset{\mathrm{j}}{a}{}^{\lambda} \ldots T^{\nu\cdots}_{\lambda\cdots}.$$

In subsequent considerations we shall use the tensor $k_{\lambda\mu} = g_{[\lambda\mu]}$ as well as the tensors $\overset{xy}{Q}{}_{\lambda\mu}$, defined by I (10.6)b. Moreover, we shall put $a = b = \sqrt{2}$ in (3.3) and use the scalars α, β defined by I (6.2).

THEOREM (8.1). *We have*

$$(8.5)\text{a} \qquad k_{\mathrm{ab}} = \mathfrak{k}_{\mathrm{ab}}$$

$$(8.5)\text{b} \qquad \overset{12}{Q}{}_{\mathrm{ab}} = \alpha^{-1}\, {}^{2}\mathscr{Q}_{\mathrm{ab}}$$

$$(8.5)\text{c} \qquad \overset{34}{Q}{}_{\mathrm{ab}} = \beta^{-1}\, {}^{1}\mathscr{Q}_{\mathrm{ab}}.$$

PROOF. One obtains from I (9.3)c$_2$ for $p=1$ and from I (10.8)a and I (10.5)

$$(8.6)\text{a} \qquad k_{\mathrm{xy}} = k_{\omega\mu} \underset{\mathrm{x}}{a}{}^{\omega} \underset{\mathrm{y}}{a}{}^{\mu} = 2[\alpha \underset{1}{\rho}\underset{2}{\rho}\, \delta^{12}_{[\mathrm{xy}]} + \beta \underset{3}{\rho}\underset{4}{\rho}\, \delta^{34}_{[\mathrm{xy}]}]$$

$$= 2(\delta^{12}_{[\mathrm{xy}]} + \delta^{34}_{[\mathrm{xy}]})$$

$$(8.6)\text{b} \qquad \overset{12}{Q}{}_{\mathrm{xy}} = \overset{12}{Q}{}_{\omega\mu} \underset{\mathrm{x}}{a}{}^{\omega} \underset{\mathrm{y}}{a}{}^{\mu} = 2 \underset{1}{\rho}\underset{2}{\rho}\, \delta^{12}_{[\mathrm{xy}]}$$

$$(8.6)\text{c} \qquad \overset{34}{Q}{}_{[\mathrm{xy}]} = \overset{34}{Q}{}_{\omega\mu} \underset{\mathrm{x}}{a}{}^{\omega} \underset{\mathrm{y}}{a}{}^{\mu} = 2 \underset{3}{\rho}\underset{4}{\rho}\, \delta^{34}_{[\mathrm{xy}]}.$$

Comparing these equations with (7.1) and (3.3) (for $a = b = \sqrt{2}$), we obtain (8.5).

THEOREM (8.2). *The equations* (8.5) *are invariant with respect to the transformations* (8.3) *in* $P_3(O)$ *as well as with respect to the transformations* (5.1), (5.7) *in* $\mathscr{S}_3(O)$.

PROOF. The components k_{ab} are transformed by (8.3) according to

(8.7)a $$'k_{\mathrm{xy}} = \sum \overset{-1}{C}_x{}^z \overset{-1}{C}_y{}^t k_{\mathrm{zt}}.[1]$$

These equations reduce by virtue of (1.11)d and (8.6)a to

(8.7)b $$'k_{\mathrm{xy}} = k_{\mathrm{xy}}.$$

The corresponding transformation in $\mathscr{S}_3(O)$ is given by (5.1), (5.7). Therefore,

(8.7)c $$'k_{\mathrm{ab}} = \overset{-1}{\mathscr{C}}_{\mathrm{a}}{}^{\mathrm{c}} \overset{-1}{\mathscr{C}}_{\mathrm{b}}{}^{\mathrm{d}} k_{\mathrm{cd}} = k_{\mathrm{ab}}.$$

We see from (8.7) that (8.5)a is invariant with respect to (8.3), and (5.1), (5.7). The invariance of the remaining equations (8.5) can be proved in the same way.

In order to derive some consequences from the previous results, we need a new conception, introduced by the following

DEFINITION (8.1). *Let $\mathscr{T}^{\mathrm{a}\cdots}_{\mathrm{b}\cdots}$ be a $\mathscr{C}$-tensor. Whenever*

$$\mathscr{T}^{\nu\cdots}_{\lambda\cdots} \overset{\mathrm{def}}{=} \mathscr{T}^{\mathrm{b}\cdots}_{\mathrm{a}\cdots} \underset{\mathrm{b}}{a}^{\nu} \overset{\mathrm{a}}{a}_{\lambda} \ldots = {}'\mathscr{T}^{\mathrm{b}\cdots}_{\mathrm{a}\cdots} {}'\underset{\mathrm{b}}{a}^{\nu} \, {}'\overset{\mathrm{a}}{a}_{\lambda} \ldots,$$

then $\mathscr{T}^{\nu\cdots}_{\lambda\cdots}$ will be referred to as an absolute tensor.

THEOREM (8.3). *The tensors*

(8.8)a $$k_{\omega\mu} \overset{\mathrm{def}}{=} k_{\mathrm{ab}} \overset{\mathrm{a}}{a}_{\omega} \overset{\mathrm{b}}{a}_{\mu}$$

(8.8)b $$1\mathscr{Q}_{\omega\mu} \overset{\mathrm{def}}{=} 1\mathscr{Q}_{\mathrm{ab}} \overset{\mathrm{a}}{a}_{\omega} \overset{\mathrm{b}}{a}_{\mu}$$

(8.8)c $$2\mathscr{Q}_{\omega\mu} \overset{\mathrm{def}}{=} 2\mathscr{Q}_{\mathrm{ab}} \overset{\mathrm{a}}{a}_{\omega} \overset{\mathrm{b}}{a}_{\mu}$$

are absolute tensors. In fact,

(8.9)a $$k_{\omega\mu} = k_{\omega\mu}$$

(8.9)b $$2\mathscr{Q}_{\omega\mu} = \alpha \overset{12}{Q}_{\omega\mu}$$

(8.9)c $$1\mathscr{Q}_{\omega\mu} = \beta \overset{34}{Q}_{\omega\mu}.$$

---

[1] The symbol Σ indicates here the summation for z, z and t, t.

PROOF. Combining (8.7)c and (7.1) we obtain by virtue of (8.3) and (1.11)d

$$'\mathfrak{k}_{ab}\,'\overset{a}{a}_{\omega}\,'\overset{b}{a}_{\mu} = \mathfrak{k}_{ab}\,\overset{a}{a}_{\omega}\,\overset{b}{a}_{\mu}.$$

Therefore, $\mathfrak{k}_{\omega\mu}$ defined by (8.8)a is an absolute tensor. In a similar manner one finds that ${}^{1}\mathcal{Q}_{\omega\mu}$ and ${}^{2}\mathcal{Q}_{\omega\mu}$ defined by (8.8)b,c are absolute tensors. Multiplying (8.5) by $\overset{a}{a}_{\omega}\overset{b}{a}_{\mu}$ one obtains (8.9).

We know from the eleventh section of the first chapter that $k_{\omega\mu}$ is a linear complex in $P_3(O)$. Moreover, it is obvious (*Cf.* I (10.8)a) that $\overset{xy}{Q}_{\omega\mu}$ is a line in $P_3(O)$. We shall use these facts in the

THEOREM (8.4). *The space $P_3(O)$ considered as a point set is represented by the complex $k_{\omega\mu}$. The quadric H considered as a point set is mapped onto a congruence in $k_{\omega\mu}$, whose axes are the lines $\overset{xy}{Q}_{\omega\mu}$.*

PROOF. The coordinates under consideration in $P_3(O)$ are homogeneous. By virtue of (8.9)a the coordinates $\mathfrak{k}_{\omega\mu}$ define the same complex as $k_{\omega\mu}$. In a similar manner we may prove that the coordinates ${}^{2}\mathcal{Q}_{\omega\mu}$ and $\overset{12}{Q}_{\omega\mu}$ (or ${}^{1}\mathcal{Q}_{\omega\mu}$ and $\overset{34}{Q}_{\omega\mu}$) define the same line. Then the proof follows easily from Theorems (7.1) and (3.1) (statement c)).

NOTE I. Theorem (8.4) describes the manner in which $P_3(O)$ might be used to obtain $\mathscr{S}_3(O)$. The complex $k_{\omega\mu}$ is at the same time a complex of $P_3(O)$ and $\mathscr{S}_3(O)$, and the congruence $\mathscr{H}$ with the axes $\overset{xy}{Q}_{\omega\mu}$ is at the same time a congruence of $P_3(O)$ and $\mathscr{S}_3(O)$. The relationship between $P_3(O)$ and $\mathscr{S}_3(O)$ is a kind of point-line relationship; every point of $P_3(O)$ has an image in $\mathscr{S}_3(O)$ consisting of a pair of lines[1] belonging to $k_{\omega\mu}$. Every line of $k_{\omega\mu}$ in $\mathscr{S}_3(O)$ has its image in $P_3(O)$ consisting of a point. A line in $\mathscr{S}_3(O)$ which does not belong to $k_{\omega\mu}$ has no image in $P_3(O)$. It is not difficult to find the image in $\mathscr{S}_3(O)$ of a line in $P_3(O)$: The rulings of H in $P_3(O)$ are mapped on points on $\overset{xy}{Q}_{\omega\mu}$ (Theorem (4.3)). Later we shall find the image in $\mathscr{S}_3(O)$ of a tangential line to H in $P_3(O)$. The image in $\mathscr{S}_3(O)$ of an arbitrary line of $P_3(O)$ is discussed in Hlavatý, V., (2), (3). The image in $P_3(O)$ of a point in $\mathscr{S}_3(O)$ will be discussed later.

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[1] These lines coincide if the point is isotropic.

NOTE II. The space $P_3(O)$ with H may be looked upon as a non-Euclidean space with the absolute H. From the above Note I we see that the complex $k_{\omega\mu}$ with the congruence $\mathscr{H}$ may be looked upon as a three-dimensional non-Euclidean line space with the absolute $\mathscr{H}$. This point of view was adopted in Hlavatý, V., (3) for introducing a non-Euclidean line metric in $\mathscr{S}_3(O)$.

NOTE III. The representation of $P_3(O)$ in $\mathscr{S}_3(O)$ requires the knowledge of the complex $k_{\omega\mu}$ and of a congruence defined by its axes $\overset{xy}{Q}_{\omega\mu}$. These elements may be constructed from the given data $g_{\lambda\mu}$. Therefore, *the tensor* $g_{\lambda\mu}$ *leads in the most natural way to the spinor algebra of the unified field theory.*

9. Representation of (6.3). In this section we shall find the representation of (6.3) in $P_3(O)$. To this effect we first prove the following

THEOREM (9.1). *Let* v^A *be an anisotropic point. Put*

(9.1)a $$\omega \overset{\text{def}}{=} h_{\lambda\mu} v^\lambda v^\mu/2 \qquad v_a \overset{\text{def}}{=} h_{ab} v^b$$

(9.1)b $$T_a{}^b \overset{\text{def}}{=} v_a v^b - \omega\delta_a^b.$$

Then the transformation

(9.1)c $$'z^b = T_a{}^b z^a, \qquad z^b = \overset{-1}{T}_a{}^b\, 'z^a$$

(*the* v^A*-reflection*) *has the following properties*:

a) *it preserves the point* v^A,

b) *it preserves pointwise the polar plane* v_a *with respect to* H *of* v^a,

c) *it preserves* H *and exchanges its reguli.*

PROOF. Put $z^a = v^a$ into (9.1)c. Then

(9.2)a $$'v^b = 2\omega v^b - \omega v^b = \omega v^b.$$

This equation proves the assertion a). Assume now that z^a is incident with the polar plane v_a, *i.e.*,

$$z^a v_a = 0.$$

Then (9.1)c reduces to

(9.2)b $$'z^b = -\omega z^b,$$

and this equation proves the assertion b). Moreover, we have from (9.1)b

$$T_a{}^b T_b{}^c = \omega^2 \delta_a^c \tag{9.3}$$

so that

$$\overset{-1}{T}_b{}^c = \omega^{-2} T_b{}^c. \tag{9.4}$$

Therefore, by virtue of (9.1)b,c,

$$'h_{ab} = \overset{-1}{T}_a{}^i \overset{-1}{T}_b{}^j h_{ij} = \omega^{-4} h_{ab}. \tag{9.2)c$$

This equation shows that H is preserved by (9.1). Let z^a be an isotropic point not incident with v_a, and let $'z^a$ be the corresponding isotropic point given by (9.1). Denote by T an arbitrary tangent to H at z^a and by $'T$ the corresponding tangent at $'z^a$. The lines T and $'T$ meet in the plane v_a (see assertion b). Thus, if T is a ruling of one of the reguli of H, $'T$ is a ruling of the other regulus of H.

THEOREM (9.2). *Let v^A be an anisotropic point. There is only one transformation*

$$'z^B = v_A{}^B z^A \tag{9.5)a$$

associated with (6.3) *for which* (5.2) *holds, namely, the transformation* (9.5)a *with the coefficient $v_A{}^B$ given by*

$$v_A{}^B = \delta^B_{\mathrm{ab}} \delta^{\mathrm{cd}}_A v_{\mathrm{c}}{}^{\mathrm{a}} v_{\mathrm{d}}{}^{\mathrm{b}}, \tag{9.5)b$$

i.e.,

$$v_A{}^i = v^i h_{Aj} v^j - \omega \delta^i_A \tag{9.5)c$$

$$\left(\begin{pmatrix} v_1{}^5 & v_2{}^5 & v_3{}^5 & v_4{}^5 & v_5{}^5 & v_6{}^5 \\ v_1{}^6 & v_2{}^6 & v_3{}^6 & v_4{}^6 & v_5{}^6 & v_6{}^6 \end{pmatrix}\right) =$$

$$\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -v_6 v^5 \\ 0 & 0 & 0 & 0 & -v_5 v^6 & 0 \end{pmatrix}\right).[1]$$

This transformation is a v^a-reflection.

PROOF. The proof of (9.5)b follows the same pattern as the

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[1] We put here

$$v_5 \overset{\mathrm{def}}{=} h_{5A} v^A = -v^5, \quad v_6 \overset{\mathrm{def}}{=} h_{6A} v^A = -v^6.$$

proof of (5.3)a. Using (1.6)b, (2.1), (9.1)b and (9.5)b we obtain

(9.6)a $$v_a{}^b = T_a{}^b$$

(9.6)b $$v_5{}^6 = v_6{}^5 = -\omega = -v_5v^6 = -v_6v^5$$

(9.6)c $$\text{the remaining } v_A{}^B = 0.$$

The equations (9.6)a show that $v_a{}^b$ is a v^a-reflection. Moreover, the equations (9.6) are equivalent to (9.5). The only thing which remains to be proved is that (9.6)b, (9.6)c are compatible with (9.6)a. If we put, in accordance with (1.6)b,

(9.7)a $$z^5 = -z^6 = -\epsilon\sqrt{\zeta}$$
$$\zeta = h_{ab}z^az^b/2,$$

then we must have

(9.7)b $$'z^5 = -'z^6 = -\epsilon\sqrt{'\zeta}$$
$$'\zeta = h_{ab}{}'z^a{}'z^b/2.$$ [1]

On the other hand, we easily obtain from (9.6)a and (9.1)b

(9.7)c $$2'\zeta = h_{ab}T_c{}^aT_d{}^bz^cz^d$$
$$= \omega^2h_{cd}z^cz^d = 2\omega^2\zeta$$

or

(9.8) $$('z^5)^2 = ('z^6)^2 = \omega^2(z^5)^2 = \omega^2(z^6)^2.$$

Squaring both members of each of the equations

$$'z^5 = -v_6v^5z^6 \qquad 'z^6 = -v_5v^6z^5$$

which one obtains from (9.6)b, c, one gets (9.8). Hence, (9.6)b and (9.6)c are compatible with (9.6)a.

DEFINITION (9.1). *The equations* (9.5)b *will be referred to as the representation of the biaxial involution* (6.3) *in* $\mathscr{S}_3(O)$ *by the* v^a-*reflection in* $P_3(O)$.

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[1] In $T_4(O)$ the scalar $\sqrt{2\zeta}$ is the length of the vector z^a, while $\sqrt{2'\zeta}$ is the length of the transformed vector $'z^a$. In other words, we measure here the vectors z^a and $'z^a$ by means of the same metric tensor h_{ab}. Therefore, $\sqrt{2'\zeta}$ must not be confused with $\sqrt{'h_{ab}{}'z^a{}'z^b}$ which is the length of the transformed vector $'z^a$ in the metric given by (9.2)c.

NOTE. We shall see the importance of this representation later in dealing with the Dirac equations and the Schrödinger wave equations.

10. Digression. We know that the representation of $P_3(O)$ considered as a point set is a line complex in $\mathscr{S}_3(O)$. We now ask about the representation in $P_3(O)$ of $\mathscr{S}_3(O)$ considered as a spinor point set. In order to answer this question we first have to introduce some auxiliary conceptions concerning the quadric H

$$h_{ab}y^a y^b = 2(y^1y^2 + y^3y^4) = 0. \tag{10.1}$$

We shall confine ourselves to the restriction

$$y^3y^4 \neq 0. \tag{10.2}$$

The quadric H may be expressed by means of two parameters $\xi^{\mathfrak{a}}$

$$y^a = y^a(\xi^{1^\cdot}, \xi^{2^\cdot}). \tag{10.3}$$

We know that H has no essential singularities. We shall assume that the parametric expression (10.3) does not have any removable singularity in the part of H where (10.2) holds. By an H-tensor we mean a tensor of the two-dimensional space H, defined in the usual way by means of transformations of parameters $\xi^{\mathfrak{a}}$. Thus, for instance.

$$y_{\mathfrak{a}}{}^a \overset{\text{def}}{=} \frac{\partial y^a}{\partial \xi^{\mathfrak{a}}} \tag{10.4)a}$$

may be looked upon as components of four covariant H-vectors $y_{\mathfrak{a}}{}^1, \ldots, y_{\mathfrak{a}}{}^4$. We have from (10.1)

$$y^a y_{\mathfrak{a}}{}^b h_{ab} = 0. \tag{10.4)b}$$

Another example of an H-tensor is the *metric H-tensor*

$$h_{(\mathfrak{a}\mathfrak{e})} = h_{\mathfrak{a}\mathfrak{e}} \overset{\text{def}}{=} h_{ab}y_{\mathfrak{a}}{}^a y_{\mathfrak{e}}{}^b. \tag{10.5}$$

The quadric H is not changed by the transformation

$$*y^a = \rho(\xi)y^a, \quad (\rho \text{ real}). \tag{10.6}$$

THEOREM (10.1). *The metric H-tensor is transformed by* (10.6) *according to*

$$*h_{\mathfrak{a}\mathfrak{e}} = \rho^2 h_{\mathfrak{a}\mathfrak{e}}. \tag{10.7}$$

PROOF. We obtain from (10.3) and (10.6)

$$(10.8) \qquad {}^*y_{\mathfrak{a}}^{\;a} = \rho(y_{\mathfrak{a}}^{\;a} + r_{\mathfrak{a}} y^a)$$

$$\left(r_{\mathfrak{a}} \overset{\text{def}}{=} \frac{\partial}{\partial \xi^{\mathfrak{a}}} \ln \rho\right).$$

Therefore, by virtue of (10.1), (10.4) and (10.5)

$${}^*h_{\mathfrak{a}\mathfrak{e}} = {}^*y_{\mathfrak{a}}^{\;a}\, {}^*y_{\mathfrak{e}}^{\;b} h_{ab} = \rho^2 h_{\mathfrak{a}\mathfrak{e}}.$$

THEOREM (10.2). *The expressions*

$$(10.9) \qquad \begin{aligned} \overset{1\cdot}{u}_{\mathfrak{a}} &\overset{\text{def}}{=} \frac{1}{\sqrt{-y^3y^4}}(y^1 y_{\mathfrak{a}}^{\;2} + y^3 y_{\mathfrak{a}}^{\;4}) \;^{1} \\ \overset{2\cdot}{u}_{\mathfrak{a}} &\overset{\text{def}}{=} \frac{1}{\sqrt{-y^3y^4}}(y^2 y_{\mathfrak{a}}^{\;1} + y^3 y_{\mathfrak{a}}^{\;4}) \end{aligned}$$

are components of two covariant H-vectors $\overset{1\cdot}{u}_{\mathfrak{a}}$ *and* $\overset{2\cdot}{u}_{\mathfrak{a}}$. *These H-vectors have the following properties:*

a) *They are linearly independent and satisfy the relation*

$$(10.10) \qquad h_{\mathfrak{a}\mathfrak{e}} = \overset{1\cdot}{u}_{\mathfrak{a}} \overset{2\cdot}{u}_{\mathfrak{e}} + \overset{1\cdot}{u}_{\mathfrak{e}} \overset{2\cdot}{u}_{\mathfrak{a}};$$

b) *they are complex conjugates;*

c) *they are not affected by the transformations* (1.11)c,d,e;

d) *they are transformed by* (10.6) *according to*

$$(10.11) \qquad {}^*\overset{\mathfrak{e}}{u}_{\mathfrak{a}} = |\rho| \overset{\mathfrak{e}}{u}_{\mathfrak{a}}.$$

PROOF. The H-vectorial character of $\overset{\mathfrak{e}}{u}_{\mathfrak{a}}$ follows directly from (10.9). The equation (10.10) follows by direct computation from (10.9), (10.1) and (10.4). According to our assumption about the nonexistence of removable singularities of the parametric equations (10.3) we have

$$(10.12) \qquad \mathrm{Det}((h_{\mathfrak{a}\mathfrak{e}})) \neq 0.$$

This condition shows that by virtue of (10.10) the H-vectors $\overset{\mathfrak{e}}{u}_{\mathfrak{a}}$ can not be linearly dependent. Their complex conjugacy follows at

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[1] We assume the point y^a to be real with real coordinates y^3, y^4.

once from (10.9) and from the fact that y^1 and y^2 are complex conjugates and y^3, y^4 are real. The transformation (1.11)c,d,e, where, of course, φ and ψ are not functions of the parameters $\xi^{\mathfrak{a}}$, do not change the expressions

$$y^3 y^4,\ y^1 y_{\mathfrak{a}}{}^2,\ y_{\mathfrak{a}}{}^1 y^2,\ y^3 y_{\mathfrak{a}}{}^4.$$

Hence, the H-vectors $\overset{\mathfrak{e}}{u}_{\mathfrak{a}}$ are invariant under (1.11)c,d,e. The equation (10.11) follows from (10.9), (10.8) and (10.1).

The H-vectors $\overset{\mathfrak{e}}{u}_{\mathfrak{a}}$ being linearly independent, we may use them as a nonholonomic frame.

Definition (10.1). *The nonholonomic components* $\overset{\mathfrak{e}}{p}$ *of a contravariant H-vector* $p^{\mathfrak{a}}$ *are defined as follows:*

$$\overset{\mathfrak{e}}{p} \overset{\text{def}}{=} p^{\mathfrak{a}}\, \overset{\mathfrak{e}}{u}_{\mathfrak{a}}. \tag{10.13}$$

Note. The H-vector $p^{\mathfrak{a}}$ also gives rise to a point

$$p^a \overset{\text{def}}{=} y_{\mathfrak{a}}{}^a p^{\mathfrak{a}} \tag{10.14}$$

which is a point on a tangent to H at y^a. In fact, we have by virtue of (10.4)

$$p^a y^b h_{ab} = 0. \tag{10.15}$$

11. Mapping of lineal elements.

In the next theorem we shall use the points y^a and p^a introduced at the end of the previous section. The isotropic point y^a is mapped on a ruling $y^{\mathfrak{ab}}$ of $\mathscr{H}$, the anisotropic point p^a is represented by two lines

$${}^{\epsilon}p^{\mathfrak{ab}} = \sqrt{2}\,\delta_A^{\mathfrak{ab}}\, p^A. \tag{11.1}$$

Theorem (11.1). *The set of points*

$$v^A = \gamma y^A + \delta p^A \tag{11.2}$$

is represented by two pencils of spinor lines

$${}^{\epsilon}\mathfrak{v}^{\mathfrak{ab}} = \gamma y^{\mathfrak{ab}} + \delta\, {}^{\epsilon}p^{\mathfrak{ab}}. \tag{11.3}$$

The vertex ${}^{\epsilon}\mathfrak{v}^{\mathfrak{a}}$ *of the pencil* ${}^{\epsilon}\mathfrak{v}^{\mathfrak{ab}}$ *is the common spinor of* $y^{\mathfrak{ab}}$ *and* ${}^{\epsilon}p^{\mathfrak{ab}}$. *Its plane is*

$${}^{\epsilon}\mathfrak{v}_{\mathfrak{b}} = \mathfrak{k}_{\mathfrak{ab}}\, {}^{\epsilon}\mathfrak{v}^{\mathfrak{a}}. \tag{11.4}$$

PROOF. The point (11.2) is represented by the spinor lines

$$(11.5) \qquad {}^{\epsilon}v^{\mathfrak{ab}} = \sqrt{2}\delta^{\mathfrak{ab}}_{A}v^{A}$$

$$= \sqrt{2}\delta^{\mathfrak{ab}}_{A}(\gamma y^{A} + \delta p^{A}).$$

This equation is equivalent to (11.3). In order to prove that (11.3) is a pencil for variable γ and δ, it is sufficient to prove that $y^{\mathfrak{ab}}$ and $p^{\mathfrak{ab}}$ meet. We have according to (10.15)

$$\tfrac{1}{2}y_{\mathfrak{ab}}p^{\mathfrak{ab}} = y^{1}p^{2} + y^{2}p^{1} + y^{4}p^{3} + y^{3}p^{4}$$

$$= h_{ab}y^{a}p^{b} = 0.$$

Hence, $y^{\mathfrak{ab}}$ and $p^{\mathfrak{ab}}$ meet in a common spinor ${}^{\epsilon}v^{\mathfrak{a}}$. The spinor lines ${}^{\epsilon}v^{\mathfrak{ab}}$ belong to the complex k (see (7.5)). Therefore, the plane of the pencil ${}^{\epsilon}v^{\mathfrak{ab}}$ is the polar plane (11.4) of ${}^{\epsilon}v^{\mathfrak{a}}$ with respect to k.

NOTE. If ${}^{\epsilon}v^{\mathfrak{a}}$ is known, then ${}^{\epsilon}v_{\mathfrak{a}}$ is uniquely determined. Therefore, we may refer to ${}^{\epsilon}v^{\mathfrak{a}}$ *as the representation of the tangent line* $y^{[a}p^{b]}$, keeping in mind that ${}^{\epsilon}v_{\mathfrak{a}}$ is defined by (11.4).

DEFINITION (11.1). *By a lineal element* $(y^{a}, v^{\mathfrak{a}})$ *we mean an isotropic point* y^{a} *(the support of the element) and a set of H-vectors* $\rho v^{\mathfrak{a}}$ $(\rho \neq 0)$ *defined at* y^{a} *(the direction of the element). If* $v^{\mathfrak{a}}$ *is (is not) a null H-vector of* $h_{\mathfrak{ae}}$, *then the corresponding lineal element will be termed an isotropic (an anisotropic) element.*

NOTE. According to this definition a lineal element is an object of H. Therefore, its corresponding object in $\mathscr{S}_3(O)$ is *the map* of the lineal element and vice versa. The main topic of this section is the investigation of mapping of lineal elements onto $\mathscr{S}_3(O)$ and also the inverse of this mapping.

Denote by ${}^{1}y^{\mathfrak{a}}$, ${}^{2}y^{\mathfrak{a}}$ the focal spinors of the line $y^{\mathfrak{ab}}$ involved in (11.3). We may assume without loss of generality that their coordinates are [1]

$$(11.6)\text{a} \qquad {}^{1}y^{\mathfrak{a}}(\sqrt{-\,y^{3}y^{1}},\ \sqrt{-\,y^{3}y^{4}},\ 0,\ 0)$$

$$ {}^{2}y^{\mathfrak{a}}(0,\ 0,\ \sqrt{y^{4}y^{3}},\ -\,\sqrt{y^{4}y^{1}}).$$

In subsequent considerations we admit only those factors of proportionality for these coordinates such that

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[1] *Cf.* (4.3).

$$(11.6)\text{b} \qquad {}^{1}y^{1}\,{}^{1}y^{2} = {}^{2}y^{3}\,{}^{2}y^{4}.$$

This condition is invariant with respect to the transformations (5.1), (5.7).

The following three Theorems will be proved simultaneously.

THEOREM (11.2)a. *Let* $(y^{a}, v^{\mathfrak{a}})$ *be an isotropic lineal element whose direction belongs to the same regulus of H as* $\underset{1}{a}{}^{[\nu}\underset{3}{a}{}^{\lambda]}$ $(\underset{1}{a}{}^{[\nu}\underset{4}{a}{}^{\lambda]})$. *Let* y^{ab} *be the map of its support* y^{a}, *and* ${}^{1}y^{\mathrm{a}}$, ${}^{2}y^{\mathrm{a}}$, *given by* (11.6), *the focal spinors of* y^{ab}. *Then this lineal element is mapped on* ${}^{1}y^{\mathrm{a}}$ $({}^{2}y^{\mathrm{a}})$.

THEOREM (11.2)b. *Let* $(y^{a}, v^{\mathfrak{a}})$ *be a real anisotropic lineal element.*[1] *This lineal element is mapped on a pair of spinors*

$$(11.7)\text{a} \qquad {}^{\epsilon}v^{\mathrm{a}} = \epsilon\sqrt{\overset{1\cdot}{v}}\;{}^{1}y^{\mathrm{a}} + \sqrt{\overset{2\cdot}{v}}\;{}^{2}y^{\mathrm{a}}.$$

Here

$$(11.7)\text{b} \qquad \overset{\mathrm{e}}{v} \overset{\mathrm{def}}{=} v^{\mathfrak{a}}\overset{\mathrm{e}}{u}_{\mathfrak{a}}$$

are the nonholonomic components of the H-vector $v^{\mathfrak{a}}$ *with respect to the frame* $\overset{\mathrm{e}}{u}_{\mathfrak{a}}$.

THEOREM (11.2)c. *Let* $(y^{a}, v^{\mathfrak{a}})$ *be a set of lineal elements with a fixed support* y^{a} *and a variable direction* $\rho v^{\mathfrak{a}}$. *This set is mapped on an involution of spinors* ${}^{\epsilon}v^{\mathrm{a}}$ *given by* (11.7)a. *The involution consists of pairs of spinors* ${}^{+}v^{\mathrm{a}}$, ${}^{-}v^{\mathrm{a}}$. *Its double spinors are* ${}^{1}y^{\mathrm{a}}$, ${}^{2}y^{\mathrm{a}}$ *given by* (11.6).

PROOF. A lineal element $(y^{a}, v^{\mathfrak{a}})$ defines a tangent line

$$(11.8) \qquad y^{[a}y_{\mathfrak{a}}{}^{b]}v^{\mathfrak{a}}$$

of H at y^{a}. If $v^{\mathfrak{a}}$ is a null H-vector, then we have according to (10.10)

$$(11.9) \qquad v^{\mathfrak{a}}v^{\mathfrak{e}}h_{\mathfrak{ae}} = 2\overset{1\cdot}{v}\overset{2\cdot}{v} = 0,$$

hence, (11.8) is a ruling of H, and Theorem (11.2)a is an immediate consequence of Theorem (4.3) (*Cf.* also Note I following that Theorem). If v^{α} is not a null H-vector, then we have the situation described in Theorem (11.1) (*Cf.* also the Note following Theorem (11.1)).

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[1] i.e., an element whose support and direction are real. We assume without loss of generality that y^{3}, y^{4} satisfying (10.2) are real and $y^{4} > 0$. The spinors ${}^{1}y^{\mathrm{a}}$ and ${}^{2}y^{\mathrm{a}}$ involved in (11.7)a have the same meaning as in the previous Theorem.

The spinor ${}^{\epsilon}v^{a}$ is the intersection of p^{ab} and y^{ab}, and therefore, it is in the set

$$(11.10)\text{a} \qquad z^{a} = \mu\, {}^{1}y^{a} + \lambda\, {}^{2}y^{a}.$$

Disregarding (11.6) for the moment, we admit more general factors of proportionality

$$(11.11) \qquad \begin{gathered} {}^{1}y^{a}(ay^{1},\ ay^{4},\ 0,\ 0) \\ {}^{2}y^{a}(0,\ 0,\ by^{3},\ -\ by^{1}) \\ ab \neq 0 \end{gathered}$$

so that

$$(11.10)\text{b} \qquad z^{a}(\mu a y^{1},\ \mu a y^{4},\ \lambda b y^{3},\ -\ \lambda b y^{1}).$$

The intersection spinor of y^{ab} and p^{ab} is given by the ratio $\lambda : \mu$ resulting from

$$(11.12) \qquad {}^{\epsilon}v_{ab}\, z^{b} = 0$$

where, according to (11.3), (2.10) and (11.2),

$$(11.13) \quad {}^{\epsilon}v_{ab}(\delta p^{6},\ \delta p^{5},\ \gamma y^{1} + \delta p^{1},\ \gamma y^{2} + \delta p^{2},\ \gamma y^{4} + \delta p^{4},\ \gamma y^{3} + \delta p^{3}).$$

All four equations (11.12) lead to

$$(11.14)\text{a} \qquad \lambda b : \mu a = (y^{1}p^{2} + y^{4}p^{3}) : -\ p^{5}y^{3}.$$

On the other hand, we have according to (10.13), (10.14), (10.4) and (10.9)

$$(11.15)\text{a} \quad \begin{aligned} y^{1}p^{2} + y^{2}p^{1} &= p^{\alpha}(y^{1}y_{\alpha}{}^{2} + y^{4}y_{\alpha}{}^{3}) \\ &= -\ p^{\alpha}(y^{2}y_{\alpha}{}^{1} + y^{3}y_{\alpha}{}^{4}) \\ &= -\ \sqrt{-\ y^{3}y^{4}}\ p^{\alpha}\overset{2\cdot}{u}_{\alpha} = -\ \sqrt{-\ y^{3}y^{4}}\ \overset{2\cdot}{p} \end{aligned}$$

$$(11.15)\text{b} \qquad -\ p^{5} = \epsilon\sqrt{\pi} \overset{\text{def}}{=} \epsilon(p^{a}p^{b}h_{ab}/2)^{\frac{1}{2}} = \epsilon(\overset{1\cdot}{p}\ \overset{2\cdot}{p})^{\frac{1}{2}}.$$

Moreover, we see from (11.2) that the H-vectors v^{α} and p^{α} must be proportional

$$(11.16) \qquad p^{\alpha} = \nu v^{\alpha}.$$

Therefore, (11.14)a reduces to

$$(11.14)\text{b} \qquad \lambda b : \mu a = -\sqrt{-y^{3}y^{4}}\,\nu\,\overset{2\cdot}{v} : \epsilon\sqrt{\nu^{2}\ \overset{1\cdot}{v}\ \overset{2\cdot}{v}}\ y^{3}.$$

According to our assumption, $v^{\mathfrak{a}}$ is not a null H-vector, so that

$$(11.17) \qquad v^{\mathfrak{a}} v^{\mathfrak{e}} h_{\mathfrak{ae}} = 2\overset{1\cdot}{v}\,\overset{2\cdot}{v} \neq 0.$$

We may assume without loss of generality

$$v > 0.$$

Thus, (11.14)b reduces to

$$(11.14)\text{c} \qquad \lambda b : \mu a = \sqrt{y^4}\sqrt{\overset{2\cdot}{v}} : \epsilon\sqrt{-y^3}\sqrt{\overset{1\cdot}{v}}.$$

If we put

$$(11.18) \qquad b = \sqrt{y^4}, \quad a = \sqrt{-y^3}$$

into (11.11), we obtain (11.6)a and

$$(11.14)\text{d} \qquad \lambda : \mu = \sqrt{\overset{2\cdot}{v}} : \epsilon\sqrt{\overset{1\cdot}{v}}.$$

Substituting from (11.14)d into (11.10)a and writing ${}^{\epsilon}\mathfrak{v}^{\mathfrak{a}}$ for the resulting spinor, we obtain (11.7)a.[1] It is obvious that Theorem (11.2)a is a special case of Theorem (11.2)b. In fact, we obtain the spinors ${}^{1}\mathfrak{y}^{\mathfrak{a}}$ and ${}^{2}\mathfrak{y}^{\mathfrak{a}}$ from (11.7)a for $\overset{2\cdot}{v} = 0$, $\overset{1\cdot}{v} \neq 0$ and $\overset{1\cdot}{v} = 0$, $\overset{2\cdot}{v} \neq 0$. $\epsilon\sqrt{\overset{1\cdot}{v}}$ and $\sqrt{\overset{2\cdot}{v}}$ may be looked upon as projective homogeneous coordinates of ${}^{\epsilon}\mathfrak{v}^{\mathfrak{a}}$ in the frame ${}^{1}\mathfrak{y}^{\mathfrak{a}}$, ${}^{2}\mathfrak{y}^{\mathfrak{a}}$. The projective homogeneous coordinates of these latter spinors in the same frame are (1, 0) and (0, 1). Therefore, we obtain for the cross ratio $({}^{+}\mathfrak{v}^{\mathfrak{a}}, {}^{-}\mathfrak{v}^{\mathfrak{a}}, {}^{1}\mathfrak{y}^{\mathfrak{a}}, {}^{2}\mathfrak{y}^{\mathfrak{a}})$ the value

$$({}^{+}\mathfrak{v}^{\mathfrak{a}}, {}^{-}\mathfrak{v}^{\mathfrak{a}}, {}^{1}\mathfrak{y}^{\mathfrak{a}}, {}^{2}\mathfrak{y}^{\mathfrak{a}}) = \frac{\begin{vmatrix} \sqrt{\overset{1\cdot}{v}} & \sqrt{\overset{2\cdot}{v}} \\ 1 & 0 \end{vmatrix}\begin{vmatrix} -\sqrt{\overset{1\cdot}{v}} & \sqrt{\overset{2\cdot}{v}} \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} \sqrt{\overset{1\cdot}{v}} & \sqrt{\overset{2\cdot}{v}} \\ 0 & 1 \end{vmatrix}\begin{vmatrix} -\sqrt{\overset{1\cdot}{v}} & \sqrt{\overset{2\cdot}{v}} \\ 1 & 0 \end{vmatrix}} = -1.$$

Thus, ${}^{+}\mathfrak{v}^{\mathfrak{a}}$, ${}^{-}\mathfrak{v}^{\mathfrak{a}}$ is a pair of spinors belonging to the involution whose double spinors are ${}^{1}\mathfrak{y}^{\mathfrak{a}}$, ${}^{2}\mathfrak{y}^{\mathfrak{a}}$.

NOTE. The spinor coordinate transformation (5.13) which results from the identical transformation of nonholonomic coordinates in $P_3(O)$ transforms the coordinates of ${}^{1}y^{\mathfrak{a}}$ and ${}^{2}y^{\mathfrak{a}}$ in such a way that

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[1] If we allow the radical sign involved in (11.7)a and (11.14)d to represent also negative square roots then these equations hold even if the lineal element under consideration is not real.

$$'{}^{1}y^{\mathfrak{a}} = {}^{1}y^{\mathfrak{a}}$$

$$'{}^{2}y^{\mathfrak{a}} = -\ {}^{2}y^{\mathfrak{a}}.$$

Therefore, if we neglect the common factor of proportionality -1 we obtain from (11.7)a

$$'{}^{+}\mathfrak{v}^{\mathfrak{a}} = {}^{-}\mathfrak{v}^{\mathfrak{a}}$$

$$'{}^{-}\mathfrak{v}^{\mathfrak{a}} = {}^{+}\mathfrak{v}^{\mathfrak{a}}.$$

Hence, every individual pair of the involution mentioned in Theorems (11.2)b,c is preserved by (5.12), as was to be expected.

THEOREM (11.3)a. *Let v^A be an anisotropic point. The set of all lineal elements common to H and the cone circumscribed to H which has v^A for its vertex is mapped on the axes ${}^{\epsilon}\mathfrak{v}^{\mathfrak{ab}}$ of the biaxial involution* (6.3).

PROOF. The point v^A is represented by the spinor lines (2.8)a which are the axes of the biaxial involution. All directions of lineal elements mentioned in Theorem (11.3)a have the point v^A in common. Therefore, each of these individual lineal elements is mapped on spinors (11.7)a incident with ${}^{\epsilon}\mathfrak{v}^{\mathfrak{ab}}$.

THEOREM (11.3)b. *The set of all lineal elements preserved by the v^A-reflection* (9.5) *is mapped on spinors which are preserved by the biaxial involution* (6.3), *i.e., on the spinors of the axes ${}^{\epsilon}\mathfrak{v}^{\mathfrak{ab}}$ of the biaxial involution* (6.3) *which is the map of the v^A-reflection* (9.5).

PROOF. It is clear from the geometrical properties of a v^A-reflection that the lineal elements mentioned in Theorem (11.3)a are the only ones preserved by the v^A-reflection, where v^A is the vertex of the corresponding cone under consideration. Moreover, the spinors incident with ${}^{\epsilon}\mathfrak{v}^{\mathfrak{ab}}$ are the only ones which are preserved by the biaxial involution (6.3). Theorem (11.3)b follows from these facts and from Theorem (11.3)a.

12. Continuation. In the previous section we mapped a lineal element on a pair of spinors (11.7). In this section we shall find the maps of physical spinors $z^{\mathfrak{a}}$ on lineal elements, assuming that

(12.1)a $$z^1 z^2 \neq 0.$$

For a physical spinor

$$\bar{z}^1 = z^3, \quad \bar{z}^2 = z^4, \tag{12.2}$$

so that the condition (12.1)a is equivalent to

$$z^3 z^4 \neq 0. \tag{12.1)b}$$

THEOREM (12.1). *A physical spinor* $z^{\mathfrak{a}}$ *is a map of a lineal element* (y^α, v^α) *whose real support* y^α *is given by*

$$\begin{aligned} y^1 &= \rho z^1 z^4, & y^2 &= \rho z^2 z^3 \\ y^3 &= -\rho z^1 z^3, & y^4 &= \rho z^2 z^4 \end{aligned} \tag{12.3}$$

where $\rho \neq 0$ *is an arbitrary factor of proportionality.*

PROOF. We see from (12.1) that $z^{\mathfrak{a}}$ is not incident with the axes (3.3) of the congruence $\mathscr{H}$. Therefore, there is only one ruling $y_{\mathfrak{ab}}$ of $\mathscr{H}$ incident with $z^{\mathfrak{a}}$, namely the intersection of the spinor planes defined by $z^{\mathfrak{a}}$ and ${}^2\mathscr{Q}^{\mathfrak{ab}}$ and by $z^{\mathfrak{a}}$ and ${}^1\mathscr{Q}^{\mathfrak{ab}}$:

$$\begin{aligned} -\mathscr{X}^1 z^2 + \mathscr{X}^2 z^1 &= 0 \\ \mathscr{X}^3 z^4 - \mathscr{X}^4 z^3 &= 0. \end{aligned} \tag{12.4}$$

The coordinates of this line are

$$y_{\mathfrak{ab}}(0, 0, z^1 z^4, z^2 z^3, z^2 z^4, -z^1 z^3). \tag{12.5}$$

Comparing (12.5) with the second equation of (2.10), we obtain (12.3) where ρ is an arbitrary factor of proportionality. Because $z^{\mathfrak{a}}$ is a physical spinor, $z^1 z^4$ and $z^2 z^3 = \bar{z}^4 \bar{z}^1$ are complex conjugates and $z^1 z^3 = z^1 \bar{z}^1$ and $z^2 z^4 = z^2 \bar{z}^2$ are real. Therefore, the point given by (12.3) is real because the ratio of its components y^ν is real.

NOTE. In subsequent considerations we shall always assume that the factor of proportionality ρ, involved in (12.3), is real. This assumption does not restrict the generality.

In order to formulate the next Theorem we need the following abbreviations:

$$z_\alpha{}^{\mathfrak{a}} \overset{\text{def}}{=} \frac{\partial}{\partial \xi^\alpha} z^{\mathfrak{a}} \tag{12.6)a}$$

$$\mathscr{Z}_\alpha{}^{\mathfrak{ab}} \overset{\text{def}}{=} z_\alpha{}^{[\mathfrak{a}} z^{\mathfrak{b}]} \tag{12.6)b}$$

$$\zeta_1 \overset{\text{def}}{=} \left(\frac{z^1 z^2}{z^3 z^4}\right)^{\frac{1}{2}}. \tag{12.6)c}$$

It follows from (12.6)c and (12.2) that the complex-conjugate number $\bar{\zeta}_1$ satisfies the relation

$$\text{(12.6)d} \qquad \zeta_2 \overset{\text{def}}{=} \bar{\zeta}_1 = 1/\zeta_1.$$

In the next Theorem we also use ${}^1\mathscr{Q}^{\mathrm{ab}}$ and ${}^2\mathscr{Q}^{\mathrm{ab}}$ defined by (3.3) for $a = b = \sqrt{2}$. Thus,

$$\text{(12.7)} \qquad {}^1\mathscr{Q}_{\mathrm{ab}} = \sqrt{2}\,\delta^6_{\mathrm{ab}}, \quad {}^2\mathscr{Q}_{\mathrm{ab}} = \sqrt{2}\,\delta^5_{\mathrm{ab}}.$$

THEOREM (12.2). *The H-vectors* (10.9) *may be expressed as follows*:

$$\text{(12.8)a} \qquad \overset{1\cdot}{u}_{\mathfrak{a}} = |\rho|\zeta_1 \mathscr{X}_{\mathfrak{a}}{}^{\mathrm{ab}}\,{}^1\mathscr{Q}_{\mathrm{ab}}$$

$$\text{(12.8)b} \qquad \overset{2\cdot}{u}_{\mathfrak{a}} = |\rho|\zeta_2 \mathscr{X}_{\mathfrak{a}}{}^{\mathrm{ab}}\,{}^2\mathscr{Q}_{\mathrm{ab}},$$

where ρ *is the factor involved in* (12.3).

PROOF. Upon substituting from (12.3) into (10.9) we obtain

$$\text{(12.9)} \qquad \overset{1\cdot}{u}_{\mathfrak{a}} = |\rho|(z_{\mathfrak{a}}{}^3 z^4 - z_{\mathfrak{a}}{}^4 z^3)\,\frac{z^1 z^2}{(z^1 z^2 z^3 z^4)^{\frac{1}{2}}}$$

and

$$\text{(12.10)} \qquad \overset{2\cdot}{u}_{\mathfrak{a}} = |\rho|(z_{\mathfrak{a}}{}^1 z^2 - z_{\mathfrak{a}}{}^2 z^1)\,\frac{z^3 z^4}{(z^1 z^2 z^3 z^4)^{\frac{1}{2}}}.$$

The equations (12.8)a,b follow easily from (12.6), (12.9) and (12.10).
In the next Theorem we shall complete the information concerning the lineal element mentioned in Theorem (12.1).

THEOREM (12.3). *The direction of the lineal element* $(y^{\mathrm{a}}, v^{\mathfrak{a}})$ *mentioned in Theorem* (12.1) *is given by the H-vector* $v^{\mathfrak{a}}$ *whose covariant components are*

$$\text{(12.11)} \qquad v_{\mathfrak{a}} = \mathscr{X}_{\mathfrak{a}}{}^{\mathrm{ab}} k_{\mathrm{ab}}.$$

Here k_{ab} *is the complex defined by* (7.1).

PROOF. The spinor z^{a} under consideration is incident with the spinor line (12.5). It satisfies an equation similar to (11.7)a

$$\text{(12.12)} \qquad z^{\mathrm{a}} = \epsilon\sqrt{\overset{1\cdot}{v}}\;{}^1y^{\mathrm{a}} + \sqrt{\overset{2\cdot}{v}}\;{}^2y^{\mathrm{a}}$$

where the focal spinors ${}^1y^{\mathrm{a}}$ and ${}^2y^{\mathrm{a}}$ are given by (11.6)a. Substituting into these latter conditions from (12.3), and omitting the common factor of proportionality, [1] we obtain

$$\text{(12.13)a} \qquad {}^1y^a(\sqrt{z^1 z^3}\, z^1 z^4,\ \sqrt{z^1 z^3}\, z^2 z^4,\ 0,\ 0,)$$

$$\text{(12.13)b} \qquad {}^2y^a(0,\ 0,\ -\sqrt{z^2 z^4}\, z^1 z^3,\ -\sqrt{z^2 z^4}\, z^1 z^4).$$

The conditions (12.12) and (12.13) yield

$$\text{(12.14)} \qquad z^a\Big(\epsilon(\overset{1\cdot}{v} z^1 z^3)^{\frac{1}{2}} z^1 z^4,\ \epsilon(\overset{1\cdot}{v} z^1 z^3)^{\frac{1}{2}} z^2 z^4,\ -(\overset{2\cdot}{v} z^2 z^4)^{\frac{1}{2}} z^1 z^3,\ -(\overset{2\cdot}{v} z^2 z^4)^{\frac{1}{2}} z^1 z^4\Big).$$

Therefore,

$$\text{(12.15)} \qquad \overset{1\cdot}{v} = \frac{1}{(z^1 z^4) z^3 z^4},\quad \overset{2\cdot}{v} = \frac{1}{(z^1 z^4) z^1 z^2}.$$

On the other hand, by virtue of (10.10)

$$\text{(12.16)} \qquad v_a = v^e h_{ae} = v^e(\overset{1\cdot}{u}_a \overset{2\cdot}{u}_e + \overset{1\cdot}{u}_e \overset{2\cdot}{u}_a) = \overset{2\cdot}{v}\, \overset{1\cdot}{u}_a + \overset{1\cdot}{v}\, \overset{2\cdot}{u}_a.$$

Using (12.8) and (12.15) we obtain

$$\text{(12.17)a} \qquad \overset{1\cdot}{v}\, \overset{2\cdot}{u}_a = |\rho| \mathscr{X}_a{}^{ab}\, {}^1\mathscr{Q}_{ab} \left(\frac{z^3 z^4}{z^1 z^2}\right)^{\frac{1}{2}} \frac{1}{z^3 z^4 z^1 z^4}$$

$$= |\rho| \mathscr{X}_a{}^{ab}\, {}^1\mathscr{Q}_{ab} \frac{1}{(z^1 z^2 z^3 z^4)^{\frac{1}{2}} z^1 z^4},$$

and in a similar way

$$\text{(12.17)b} \qquad \overset{2\cdot}{v}\, \overset{1\cdot}{u}_a = |\rho| \mathscr{X}_a{}^{ab}\, {}^2\mathscr{Q}_{ab} \frac{1}{(z^1 z^2 z^3 z^4)^{\frac{1}{2}} z^1 z^4}.$$

Substituting from (12.17) into (12.16), and disregarding the common factor [2]

$$\frac{|\rho|}{(z^1 z^2 z^3 z^4)^{\frac{1}{2}} z^1 z^4},$$

we obtain by virtue of (7.1) and (12.7)

$$v_a = \mathscr{X}_a{}^{ab}({}^1\mathscr{Q}_{ab} + {}^2\mathscr{Q}_{ab}) = \mathscr{X}_a{}^{ab} k_{ab}.$$

---

[1] This is allowed by virtue of (11.6)b.

[2] According to Definition (11.1) of a lineal element (y^a, v^a), it is irrelevant whether we take an H-vector v^a or φv^a where $\varphi \neq 0$ is an arbitrary factor.

13. Auxiliary Theorems. In the next section we shall apply the preceding theory to Dirac's equations and Schrödinger's equations. It is convenient for this purpose to derive first some geometrical consequences of previous results.

We associate with every point v^A in $P_3(O)$ another point ${}_*v^A$ according to the rule

$$\text{(13.1)} \qquad \begin{aligned} {}_*v^1 &= \sqrt{2}v^1, & {}_*v^3 &= -\sqrt{2}v^4, & {}_*v^5 &= \sqrt{2}v^5 \\ {}_*v^2 &= \sqrt{2}v^2, & {}_*v^4 &= -\sqrt{2}v^3, & {}_*v^6 &= \sqrt{2}v^6. \end{aligned}$$

Then we have by virtue of (1.9) and (1.6)b

$$\text{(13.2)a} \qquad {}_*v^A\,{}_*v^B h_{AB} = 0$$

$$\text{(13.2)b} \qquad {}_*\omega \overset{\text{def}}{=} h_{ab}\,{}_*v^a\,{}_*v^b/2 = v^a v^b h_{ab} = 2\omega$$

$$\text{(13.2)c} \qquad \begin{aligned} {}_*v^5 = -{}_*v^6 &= -\epsilon\sqrt{{}_*\omega} \\ = -\epsilon\sqrt{2\omega} &= \sqrt{2}v^5 = -\sqrt{2}v^6. \end{aligned}$$

If $a_{\mathfrak{j}}^{\mathfrak{i}}$, $b_{\mathfrak{j}}^{\mathfrak{i}}$, $c_{\mathfrak{j}}^{\mathfrak{i}}$ are three tensors in $\mathscr{S}_3(O)$, we denote their product by

$$\text{(13.3)} \qquad (abc)_{\mathfrak{a}}^{\mathfrak{b}} \overset{\text{def}}{=} a_{\mathfrak{a}}^{\mathfrak{i}} b_{\mathfrak{i}}^{\mathfrak{j}} c_{\mathfrak{j}}^{\mathfrak{b}}.$$

In subsequent considerations we shall need the tensors $s_{\mathfrak{a}}^{\mathfrak{b}}$, $\overset{-1}{s}{}_{\mathfrak{a}}^{\mathfrak{b}}$ which satisfy the condition

$$\text{(13.4)} \qquad s_{\mathfrak{a}}^{\mathfrak{b}} \overset{-1}{s}{}_{\mathfrak{b}}^{\mathfrak{c}} = \delta_{\mathfrak{a}}^{\mathfrak{c}}$$

and are defined as follows:

(13.5)[1]
$$((s_{\mathfrak{a}}^{\mathfrak{b}})) = \left(\left(\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{matrix}\right)\right), \quad ((\overset{-1}{s}{}_{\mathfrak{a}}^{\mathfrak{b}})) = \left(\left(\begin{matrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}\right)\right).$$

THEOREM (13.1). *Put*

$$\text{(13.6)a} \qquad {}_*v_{\mathfrak{a}}^{\mathfrak{b}} \overset{\text{def}}{=} \sqrt{2}(s\,v\,\overset{-1}{s})_{\mathfrak{a}}^{\mathfrak{b}}.$$

Then

$$\text{(13.6)b} \qquad {}_*v_{\mathfrak{a}}^{\mathfrak{b}} = -\sqrt{2}i\left(\left(\begin{matrix} 0 & 0 & -v^1 & v^4 \\ 0 & 0 & v^3 & v^2 \\ v^2 & -v^4 & 0 & 0 \\ -v^3 & -v^1 & 0 & 0 \end{matrix}\right)\right)$$

---

[1] The superscripts (subscripts) denote the rows (columns.).

$$= -i\left(\begin{pmatrix} 0 & 0 & -{}_*v^1 & -{}_*v^3 \\ 0 & 0 & -{}_*v^4 & {}_*v^2 \\ {}_*v^2 & {}_*v^3 & 0 & 0 \\ {}_*v^4 & -{}_*v^1 & 0 & 0 \end{pmatrix}\right)$$

and

$$(13.7) \qquad {}_*v_a{}^b\, {}_*v_b{}^c = {}_*\omega\delta^c_a.$$

PROOF. (13.6)b follows by straightforward calculation from (13.6)a, (13.5) and (13.1). We have by virtue of (2.2)b, (13.6)a and (13.2)b

$$\begin{aligned} {}_*v_a{}^b\, {}_*v_b{}^c &= 2s_a{}^i\, v_i{}^j\, \overset{-1}{s}_j{}^b\, s_b{}^e\, v_e{}^f\, \overset{-1}{s}_f{}^c \\ &= 2\omega\delta^c_a = {}_*\omega\delta^c_a. \end{aligned}$$

THEOREM (13.2). *The projectivity*

$$(13.8) \qquad {}'\mathscr{X}^b = {}_*v_a{}^b\, \mathscr{X}^a$$

is a biaxial involution. Its axes are

$$(13.9) \qquad {}^{\epsilon}{}_*v^{ab}(i\, {}_*v^5,\ -i\, {}_*v^6,\ {}_*v^2,\ {}_*v^1,\ {}_*v^3,\ {}_*v^4).$$

They are rulings of the complex $\mathscr{K}_{ab}$ *defined by* (7.2) *and are conjugate polars with respect to the complex* k_{ab} *defined by* (7.1).

PROOF. Multiply (13.8) by ${}_*v_b{}^c$. Then, according to (13.7),

$${}'\mathscr{X}^b\, {}_*v_b{}^c = {}_*\omega\, \mathscr{X}^c.$$

Hence, (13.8) is an involution. Its eigenspinors, *i.e.*, double spinors are given by

$$(13.10)\text{a} \qquad \lambda\mathscr{X}^b = {}_*v_a{}^b\, \mathscr{X}^a$$

where the eigenvalue λ has to be found. Multiplying this equation by ${}_*v_b{}^c$ we obtain by virtue of (13.7)

$$(13.11) \qquad \lambda^2 = {}_*\omega,$$

so that the eigenspinors are given by

$$(13.10)\text{b} \qquad \epsilon\sqrt{{}_*\omega}\, \mathscr{X}^b - {}_*v_a{}^b\, \mathscr{X}^a = 0.$$

This system of four equations is of rank 2, and therefore, its solution is a spinor line, namely ${}^{\epsilon}{}_*v^{ab}$ given by (13.9). This line is a ruling of $\mathscr{K}_{ab}$ because

$$\mathscr{K}_{ab}\, {}^{\epsilon}{}_*v^{ab} = 0.$$

The remaining statements of Theorem (13.2) can be proved in the manner used earlier for the proof of corresponding statements concerning $v_{\mathrm{a}}{}^{\mathrm{b}}$ and ${}^{\epsilon}v^{\mathrm{ab}}$.

NOTE. From (13.9) we can derive the representation ${}_*v^A \leftrightarrow {}^{\epsilon}{}_*v^{\mathrm{ab}}$ in the form

(13.12) $$\sqrt{2}\,{}_*v^A = {}_*\delta^A_{\mathrm{ab}}\,{}^{\epsilon}{}_*v^{\mathrm{ab}}, \quad {}^{\epsilon}{}_*v^{\mathrm{ab}} = \sqrt{2}\,\delta^{\mathrm{ab}}_A\,{}_*v^A,$$

where

(13.13)a $${}_*\delta^c_{\mathrm{ab}} \overset{\mathrm{def}}{=} \delta^c_{\mathrm{ab}}, \quad {}_*\delta^5_{\mathrm{ab}} \overset{\mathrm{def}}{=} -\,i\delta^5_{\mathrm{ab}}, \quad {}_*\delta^6_{\mathrm{ab}} = i\delta^6_{\mathrm{ab}},$$

(13.13)b $${}_*\delta^{\mathrm{ab}}_c \overset{\mathrm{def}}{=} \delta^{\mathrm{ab}}_c, \quad {}_*\delta^{\mathrm{ab}}_5 = i\delta^{\mathrm{ab}}_5, \quad {}_*\delta^{\mathrm{ab}}_6 = -\,i\delta^{\mathrm{ab}}_6,$$

(13.13)c $${}_*\delta^A_{\mathrm{ab}}\,{}_*\delta^{\mathrm{ab}}_B = \delta^A_B, \quad {}_*\delta^A_{\mathrm{ab}}\,{}_*\delta^{\mathrm{cd}}_A = \delta^{\mathrm{cd}}_{[\mathrm{ab}]}.$$

The transformation

(13.14)a $${}'z^b = {}_*v_a{}^b z^a$$

will be referred to as the representation in $P_3(O)$ of (13.8) if, by virtue of (13.8) and (13.14)a,

(13.15) $${}'{}_*\delta^A_{\mathrm{ab}} = {}_*\delta^A_{\mathrm{ab}}.$$

THEOREM (13.3). *The representation* (13.14)a of (13.8) *is a* ${}_*v^{\mathrm{a}}$*-reflection*

(13.14)b $${}_*v_a{}^b = {}_*v_a\,{}_*v^b - {}_*\omega\delta^{\mathrm{b}}_{\mathrm{a}}$$
$$({}_*v_a \overset{\mathrm{def}}{=} h_{ab}\,{}_*v^b).$$

PROOF. Following the pattern by which we derived (9.5)b, we obtain

(13.14)c $${}_*v_a{}^b = {}_*\delta^b_{\mathrm{ij}}\,{}_*\delta^{\mathrm{ef}}_a\,{}_*v_{\mathrm{e}}{}^{\mathrm{i}}\,{}_*v_{\mathrm{f}}{}^{\mathrm{j}}.$$

Substituting into these equations from (13.6)b, we obtain (13.14)b. Comparing (13.14)b with (9.1)b, we see that (13.14) is a ${}_*v^a$-reflection.

NOTE I. The eigenpoints, *i.e.*, the double points, of (13.14) are given by

(13.16)a $$\rho z^b = {}_*v_a{}^b z^a$$

where ρ is to be found. From (9.3) applied to our case we see that, if we multiply (13.16)a by ${}_{*}v_b{}^c$, we obtain

$$\text{(13.16)b} \qquad \rho^2 = {}_{*}\omega^2.$$

Hence, the eigenpoints are given by

$$\text{(13.17)} \qquad \pm\, {}_{*}\omega z^b = {}_{*}v_a{}^b z^a.$$

From (9.2)a,b applied to our case we see that the upper sign refers to the point ${}_{*}v^a$, while the lower sign refers to the points of the plane ${}_{*}v_a$. In particular, the lower sign also refers to the supports of lineal elements common to H and to the cone circumscribed to H from ${}_{*}v^a$. Comparing (13.11) and (13.16) we obtain for these supports

$$\text{(13.18)} \qquad \rho = -\lambda^2 = -\,{}_{*}\omega.$$

NOTE II. A Theorem similar to Theorem (11.3)b can also be proved here.

14. Dirac's equations and Schrödinger's equations.

In this section we shall apply previous results to the Dirac equations and the Schrödinger equations. We introduce them in the form in which they appear in van der Waerden, B. L., (1) (in a slightly different notation). The coordinates will be designated by x, y, z, t, and the metric form is assumed to be in the form

$$\text{(14.1)} \qquad x^2 + y^2 + z^2 - c^2t^2$$

where c is the velocity of light propagation. We shall also need the Pauli matrices[1]

---

[1] These matrices play, in the geometry of "two-component spinors", the role displayed by $E_{\lambda\mathrm{a}}{}^{\mathrm{b}}$ (see eq. (1.14)) in the geometry of "four-component spinors". In fact, they may easily be expressed in terms of $E_{\lambda\mathrm{a}}{}^{\mathrm{b}}$. Defining the indices $\dot{1}$ and $\dot{2}$ by $\dot{1} \overset{\mathrm{def}}{=} 3$, $\dot{2} \overset{\mathrm{def}}{=} 4$, we have

$$\begin{aligned} E_{\lambda\mathrm{a}}{}^{\dot{\mathrm{b}}}(\underset{3}{a^\lambda} + \underset{4}{a^\lambda}) &= \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \\ iE_{\lambda\mathrm{a}}{}^{\dot{\mathrm{b}}}(-\underset{3}{a^\lambda} + \underset{4}{a^\lambda}) &= \left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) \\ E_{\lambda\mathrm{a}}{}^{\dot{\mathrm{b}}}(\underset{2}{a^\lambda} + \underset{1}{a^\lambda}) &= \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\ E_{\lambda\mathrm{a}}{}^{\dot{\mathrm{b}}}(\underset{2}{a^\lambda} - \underset{1}{a^\lambda}) &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right). \end{aligned} \qquad \mathrm{a} = 1, 2$$

In two-component spinor geometry, besides (14.2), one has to use the

$$\sigma_x \overset{\text{def}}{=} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \quad \sigma_y \overset{\text{def}}{=} \left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) \tag{14.2}$$

$$\sigma_z \overset{\text{def}}{=} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \quad \sigma_t \overset{\text{def}}{=} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

and the matrices

$$\mathscr{X}_{\cdot} \overset{\text{def}}{=} \left(\begin{pmatrix} \mathscr{X}_1 \\ \mathscr{X}_2 \end{pmatrix}\right) \tag{14.3}$$

$$\mathscr{X}^{\cdot} \overset{\text{def}}{=} \left(\begin{pmatrix} \mathscr{X}^3 \\ \mathscr{X}^4 \end{pmatrix}\right). \tag{14.4}$$ [1]

The components of the linear operator involved in Dirac's equations will be denoted by

$$\begin{aligned} d_x &\overset{\text{def}}{=} \frac{\hbar}{i} \frac{\partial}{\partial x} + A_x \\ &\vdots \\ d_t &\overset{\text{def}}{=} \frac{\hbar}{i} \frac{\partial}{\partial t} + A_t. \end{aligned} \tag{14.5a}$$

Here $\hbar$ is Planck's constant divided by 2π and $A(A_x, A_y, A_z, A_t)$ is a

matrices

$$\sigma_x{}' \overset{\text{def}}{=} -\sigma_x, \quad \sigma_y{}' \overset{\text{def}}{=} -\sigma_y, \quad \sigma_z{}' \overset{\text{def}}{=} -\sigma_z, \quad \sigma_t{}' \overset{\text{def}}{=} \sigma_t.$$

These matrices are given by

$$\begin{aligned} E_{\lambda\dot{\mathfrak{a}}}{}^{\mathfrak{b}}(-\underset{3}{a^\lambda} - \underset{4}{a^\lambda}) &= \left(\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right) \\ iE_{\lambda\dot{\mathfrak{a}}}{}^{\mathfrak{b}}(\underset{3}{a^\lambda} - \underset{4}{a^\lambda}) &= \left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right) \qquad \mathfrak{b} = 1, 2 \\ E_{\lambda\dot{\mathfrak{a}}}{}^{\mathfrak{b}}(-\underset{1}{a^\lambda} - \underset{2}{a^\lambda}) &= \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ E_{\lambda\dot{\mathfrak{a}}}{}^{\mathfrak{b}}(\underset{1}{a^\lambda} - \underset{2}{a^\lambda}) &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

In v.d. Waerden, (1), one finds the notation $\sigma_{k\lambda\dot{\mu}}$, $\sigma_k'^{\dot{\lambda}\mu}$ for the components of these matrices.

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[1] According to (3.10)b we have for a *physical* spinor

$$\mathscr{X}^{\cdot} = \left(\begin{pmatrix} \bar{\mathscr{X}}^1 \\ \bar{\mathscr{X}}^2 \end{pmatrix}\right).$$

vector built up from the electric potential, the magnetic potential and the charge of the electron.

The Dirac equations may be written

$$(14.5)\text{b} \qquad \begin{aligned} &(d_x\sigma_x + d_y\sigma_y + d_z\sigma_z + c^{-1}d_t)\ \mathscr{X}_{\cdot} + \mu c\ \mathscr{X}^{\cdot} = 0 \\ &(-\ d_x\sigma_x - d_y\sigma_y - d_z\sigma_z + c^{-1}d_t)\ \mathscr{X}^{\cdot} + \mu c\ \mathscr{X}_{\cdot} = 0. \end{aligned}$$

Here μ is the mass of the electron. The corresponding Schrödinger relativistic wave equation is

$$(14.6) \qquad (d_x{}^2 + d_y{}^2 + d_z{}^2 - c^{-2}d_t{}^2)\zeta + \mu^2c^2\zeta = 0.$$

In the next Theorem we shall use the contravariant components (with respect to the metric (14.1))of the linear operator (d_x, d_y, d_z, d_t) used in (14.5)b

$$(14.7)\text{a} \qquad d^x \overset{\text{def}}{=} d_x,\ d^y \overset{\text{def}}{=} d_y,\ d^z \overset{\text{def}}{=} d_z,\ d^t \overset{\text{def}}{=} -\ c^{-2}d_t.$$

THEOREM (14.1). *Introduce the symbolic point*

$$(14.7)\text{b} \qquad \begin{aligned} &\sqrt{2}v^1 \overset{\text{def}}{=} d^x + id^y \qquad &&\sqrt{2}v^2 \overset{\text{def}}{=} d^x - id^y \\ &\sqrt{2}v^3 \overset{\text{def}}{=} d^z + cd^t \qquad &&\sqrt{2}v^4 \overset{\text{def}}{=} d^z - cd^t, \end{aligned}$$

and define ${}_*v_{\mathfrak{a}}{}^{\mathfrak{b}}$ *by means of* (13.6)a. *Then the Dirac equations* (14.5)b *acquire the form*

$$(14.8) \qquad {}_*v_{\mathfrak{a}}{}^{\mathfrak{b}}\,\mathscr{X}^{\mathfrak{a}} + i\mu c\mathscr{X}^{\mathfrak{b}} = 0.$$

PROOF. The Dirac equations (14.5)b may be condensed into the single equation

$$(14.9)\text{a} \quad \sqrt{2}\left(\begin{pmatrix} 0 & 0 & -v^3 & -v^2 \\ 0 & 0 & -v^1 & v^4 \\ v^4 & v^2 & 0 & 0 \\ v^1 & -v^3 & 0 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} \mathscr{X}_1 \\ \mathscr{X}_2 \\ \mathscr{X}^3 \\ \mathscr{X}^4 \end{pmatrix}\right) + \mu c\left(\begin{pmatrix} \mathscr{X}_1 \\ \mathscr{X}_2 \\ \mathscr{X}^3 \\ \mathscr{X}^4 \end{pmatrix}\right) = 0.$$

On the other hand, we have

$$(14.9)\text{b} \quad \left(\begin{pmatrix} \mathscr{X}_1 \\ \mathscr{X}_2 \\ \mathscr{X}^3 \\ \mathscr{X}^4 \end{pmatrix}\right) = \left(\begin{pmatrix} \mathscr{X}^2 \\ -\ \mathscr{X}^1 \\ \mathscr{X}^3 \\ \mathscr{X}^4 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} \mathscr{X}^1 \\ \mathscr{X}^2 \\ \mathscr{X}^3 \\ \mathscr{X}^4 \end{pmatrix}\right).$$

Denote by $\mathscr{P}_{\mathfrak{a}}{}^{\mathfrak{b}}$ the elements of the 4 × 4 matrix in (14.9)b and by $\sqrt{2}\mathscr{V}_{\mathfrak{a}}{}^{\mathfrak{b}}$ the elements of the 4 × 4 matrix in (14.9)a. Then we may write for (14.9)a, according to (14.9)b,

$$\sqrt{2}\,\mathscr{V}_{a}{}^{b}\mathscr{P}_{c}{}^{a}\mathscr{X}^{c} + \mu c\,\mathscr{P}_{c}{}^{b}\mathscr{X}^{c} = 0,$$

and this equation is equivalent to

$$\overset{-1}{\mathscr{P}}_{b}{}^{d}\sqrt{2}\,\mathscr{V}_{a}{}^{b}\mathscr{P}_{c}{}^{a}\mathscr{X}^{c} + \mu c\,\mathscr{X}^{d} = 0$$

or

$$\text{(14.9)c} \qquad \sqrt{2}(\mathscr{P}\mathscr{V}\overset{-1}{\mathscr{P}})_{c}{}^{d}\mathscr{X}^{c} + \mu c\,\mathscr{X}^{d} = 0$$

where $\overset{-1}{\mathscr{P}}_{a}{}^{b}$ is inverse to $\mathscr{P}_{a}{}^{b}$

$$((\overset{-1}{\mathscr{P}}_{a}{}^{b})) = \left(\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right).$$

A straightforward computation shows that by virtue of (13.6)b and (13.1)

$$\text{(14.10)} \qquad \sqrt{2}((\,(\mathscr{P}\mathscr{V}\overset{-1}{\mathscr{P}})_{c}{}^{d}))$$

$$= \sqrt{2}\left(\begin{pmatrix} 0 & 0 & v^1 & -v^4 \\ 0 & 0 & -v^3 & -v^2 \\ -v^2 & v^4 & 0 & 0 \\ v^3 & v^1 & 0 & 0 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 & 0 & {}_*v^1 & {}_*v^3 \\ 0 & 0 & {}_*v^4 & -{}_*v^2 \\ -{}_*v^2 & -{}_*v^3 & 0 & 0 \\ -{}_*v^4 & {}_*v^1 & 0 & 0 \end{pmatrix}\right)$$

$$= -\,i((\,{}_*v_{a}{}^{b})).$$

Substituting from (14.10) into (14.9)c, and multiplying the resulting equation by i, we obtain (14.8).

THEOREM (14.2). *Dirac's equations* (14.8) *define eigenspinors of the symbolical projectivity* (13.8). *These eigenspinors also satisfy the differential equations*

$$\text{(14.11)} \qquad ({}_*\omega + \mu^2c^2)\mathscr{X}^{a} = 0,$$

where ${}_*\omega$ *is defined by* (13.2)b *and* (14.7)b.

PROOF. One obtains this theorem by comparing (14.8) and (13.10)a, (13.11).

NOTE. Equations (14.8) define spinors incident with one of the symbolical axes $\epsilon_{{}_*v}{}^{ab}$ of the symbolical projectivity (13.8). It is clear that the equations

$$ {}_{*}v_{\mathbf{a}}{}^{\mathbf{b}}\, \mathscr{X}^{\mathbf{a}} - i\mu c\, \mathscr{X}^{\mathbf{b}} = 0, \tag{14.12} $$

which define the spinors on the remaining axis, must also have physical meaning.

We know [1] that the eigenspinors of the biaxial involution (13.8) are mapped onto the lineal elements common to H and to the cone circumscribed to H from the point ${}_{*}v^{a}$ which is represented by the axes ${}^{\epsilon}{}_{*}v^{\mathbf{ab}}$ of the biaxial involution. These lineal elements of H are preserved by the ${}_{*}v^{a}$-reflection (13.14) which is the representation of (13.8). We shall refer to them as *double lineal elements* of (13.14).

THEOREM (14.3). *Let* z^{a} *be the supports of double lineal elements of* (13.14) *which map in* $P_3(O)$ *the eigenspinors of Dirac's equations* (14.8). *Put* $\zeta = \zeta\,(z^{a})$ *and assume that* ζ *is transformed by* (13.14) *in the same manner as*

$$ z^1z^2 + z^3z^4. \tag{14.13} $$

Then ζ *satisfies the Schrödinger wave equation* (14.6).

PROOF. Disregard for a moment (14.7) and consider the relationship between the double elements of (13.8) and (13.14). We see from (14.8) (and (14.12)) that the eigenvalues λ of the biaxial involution (13.8) are given by

$$ \lambda^2 = (i\mu c)^2 = -\,\mu^2c^2. \tag{14.14} $$

If we apply (9.7)c to our case (13.14), we obtain according to our assumption

$$ '\zeta = {}_{*}\omega^2\zeta. \tag{14.15)a} $$

In particular, if $'z^{a}$ is a double point of (13.14)

$$ 'z^{a} = \rho z^{a}, $$

the equation (14.15)a splits into two

$$ (\rho + {}_{*}\omega)\zeta = 0 \tag{14.15)b} $$

$$ (\rho - {}_{*}\omega)\zeta = 0. \tag{14.15)c} $$

If we multiply the equation for double points

$$ \rho z^{b} = {}_{*}v_{a}{}^{b}z^{a} \tag{14.16)a} $$

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[1] *Cf. Note II* at the end of section 13.

by ${}_*v_b{}^c$, we obtain by virtue of (13.16)b two equations

$$(\rho + {}_*\omega)z^a = 0 \tag{14.16)b}$$

$$(\rho - {}_*\omega)z^a = 0. \tag{14.16)c}$$

From the discussion of the meaning of equation (13.17) we see that (14.15)c and (14.16)c refer to the isolated double point ${}_*v^a$ of (13.14), while (14.15)b and (14.16)b refer to the double points of (13.14) in the plane ${}_*v_a = {}_*v^b h_{ab}$. Therefore, they also refer to the supports mentioned in our Theorem. Using (13.18) and (14.14) we may write for (14.15)b

$$({}_*\omega + \mu^2 c^2)\zeta = 0. \tag{14.17}$$

We shall transform this equation by means of (14.7)a,b. These equations, together with (13.1) and (13.2)b, lead to

$$\begin{aligned} {}_*\omega = 2\omega = h_{ab}v^a v^b &= 2(v^1 v^2 + v^3 v^4) \\ &= (d^x)^2 + (d^y)^2 + (d^z)^2 - c^2(d^t)^2 \\ &= (d_x)^2 + (d_y)^2 + (d_z)^2 - c^{-2}(d_t)^2. \end{aligned}$$

Substituting from this equation into (14.17), one obtains the Schrödinger wave equations (14.6).

II. *Spinor space $\mathscr{S}_3$ and its connection*

15. Projection of $\mathscr{S}_3(U)$ onto $\mathscr{S}_3(O)$. Denote by O an arbitrary but fixed point of space-time and by U its generic point. The algebraic theory of $\mathscr{S}_3(O)$ as developed in the first part of this chapter may also be applied to the local spinor space $\mathscr{S}_3(U)$ associate with U. Then, of course, all objects of $\mathscr{S}_3(U)$ are expressed in terms of coordinates x^ν of the point U. Thus, for instance, Φ defined by (5.7)c is a function of position, $\Phi = \Phi(U)$, and so are the coefficients $\mathscr{C}_{\mathrm{a}}{}^{\mathrm{b}}$ and $\overset{-1}{\mathscr{C}}_{\mathrm{a}}{}^{\mathrm{b}}$ defined by means of (5.7)a,b. Among the objects of $\mathscr{S}_3(U)$ there are some whose homogeneous projective coordinates are the same as the homogeneous projective coordinates of the corresponding objects in $\mathscr{S}_3(O)$. Thus, for instance, the axes ${}^1\mathscr{Q}^{\mathrm{ab}}(O)$ and ${}^2\mathscr{Q}^{\mathrm{ab}}(O)$ of the congruence $\mathscr{H}(O)$ are given by (3.3), while the axes ${}^1\mathscr{Q}^{\mathrm{ab}}(U)$ and ${}^2\mathscr{Q}^{\mathrm{ab}}(U)$ of $\mathscr{H}(U)$ in $\mathscr{S}_3(U)$ are given by

$${}^1\mathscr{Q}^{\mathrm{ab}}(U) = a(U)\delta_5^{\mathrm{ab}},\ {}^2\mathscr{Q}^{\mathrm{ab}}(U) = b(U)\delta_6^{\mathrm{ab}} \tag{15.1}$$

and, therefore, have the same homogeneous coordinates in $\mathscr{S}_3(U)$ as ${}^1\mathscr{Q}^{ab}(O)$, ${}^2\mathscr{Q}^{ab}(O)$ have in $\mathscr{S}_3(O)$. Therefore, we may superimpose $\mathscr{S}_3(U)$ onto $\mathscr{S}_3(O)$ by identifying ${}^1\mathscr{Q}^{ab}(U)$ with ${}^1\mathscr{Q}^{ab}(O)$ and ${}^2\mathscr{Q}^{ab}(U)$ with ${}^2\mathscr{Q}^{ab}(O)$. This superimposition will be referred to as *a projection of* $\mathscr{S}_3(U)$ onto $\mathscr{S}_3(O)$. It is defined as follows:

DEFINITION (15.1). *The projection of* $\mathscr{S}_3(U)$ *onto* $\mathscr{S}_3(O)$ *is a projectivity defined by coefficients of the type*

$$\mathscr{P}_a{}^b \overset{\text{def}}{=} \mathscr{P}_a{}^b(U, O) \tag{15.2}$$

of class C_2. *This projectivity has the following properties:*

a) *It reduces to an identity for* $U = O$

$$\mathscr{P}_a{}^b(O, O) = \delta_a^b. \tag{15.3}$$

b) *It carries the axes* ${}^1\mathscr{Q}^{ab}(U)$, ${}^2\mathscr{Q}^{ab}(U)$ *into* ${}^1\mathscr{Q}^{ab}(O)$, ${}^2\mathscr{Q}^{ab}(O)$.

c) *It carries any physical spinor of* $\mathscr{S}_3(U)$ *into a physical spinor of* $\mathscr{S}_3(O)$.

In the next Theorem we assume that U is sufficiently near to O, *i.e.*, $x^\nu \to 0$, where x^ν are the coordinates of U and the coordinates of O are assumed to be equal to zero.

THEOREM (15.1). *The coefficients* $\mathscr{P}_a{}^b$ *satisfy the conditions*

$$((\mathscr{P}_a{}^b)) = \left(\left(\begin{matrix} \mathscr{P}_1{}^1 & \mathscr{P}_2{}^1 & 0 & 0 \\ \mathscr{P}_1{}^2 & \mathscr{P}_2{}^2 & 0 & 0 \\ 0 & 0 & \bar{\mathscr{P}}_1{}^1 & \bar{\mathscr{P}}_2{}^1 \\ 0 & 0 & \bar{\mathscr{P}}_1{}^2 & \bar{\mathscr{P}}_2{}^2 \end{matrix}\right)\right) \tag{15.4)a}$$

$$\mathscr{P}_a{}^b(U, O) = \delta_a^b + x^\lambda(\partial_\lambda \mathscr{P}_a{}^b(U, O))_O + \dots . \tag{15.4)b}$$

PROOF. The coefficients are of class C_2, and therefore, by virtue of (15.3) they must satisfy (15.4)b. Denote by $z^a(U)$ an arbitrary spinor in $\mathscr{S}_3(U)$, and by $z^a(U|O)$ its image in $\mathscr{S}_3(O)$:

$$z^a(U|O) = \mathscr{P}_b{}^a(U, O)z^b(U). \tag{15.5}$$

Let $y^a(U)$ ($t^a(U)$) be a spinor incident with ${}^1\mathscr{Q}^{ab}(U)$ (${}^2\mathscr{Q}^{ab}(U)$) in $\mathscr{S}_3(U)$:

$$y^3(U) = y^4(U) = 0 \tag{15.6)a}$$

$$t^1(U) = t^2(U) = 0.$$

According to requirement b) of Definition (15.1) we must have

$$(15.6)\text{b} \qquad \begin{aligned} y^3(U|O) &= y^4(U|O) = 0 \\ t^1(U|O) &= t^2(U|O) = 0. \end{aligned}$$

The equations (15.6) yield

$$(15.7)\text{a} \qquad \left(\begin{pmatrix} \mathscr{P}_3{}^1 & \mathscr{P}_4{}^1 \\ \mathscr{P}_3{}^2 & \mathscr{P}_4{}^2 \end{pmatrix}\right) = \left(\begin{pmatrix} \mathscr{P}_1{}^3 & \mathscr{P}_2{}^3 \\ \mathscr{P}_1{}^4 & \mathscr{P}_2{}^4 \end{pmatrix}\right) = ((0)).$$

If $z^{\mathbf{a}}(U)$ is a physical spinor

$$(15.8)\text{a} \qquad z^3(U) = \bar{z}^1(U), \quad z^4(U) = \bar{z}^2(U),$$

we must have according to requirement c) of Definition (15.1)

$$(15.8)\text{b} \qquad z^3(U|O) = \bar{z}^1(U|O), \quad z^4(U|O) = \bar{z}^2(U|O).$$

The equations (15.5), (15.7)a and (15.8)a,b lead to

$$(15.7)\text{b} \qquad \left(\begin{pmatrix} \mathscr{P}_3{}^3 & \mathscr{P}_4{}^3 \\ \mathscr{P}_3{}^4 & \mathscr{P}_4{}^4 \end{pmatrix}\right) = \left(\begin{pmatrix} \overline{\mathscr{P}}_1{}^1 & \overline{\mathscr{P}}_2{}^1 \\ \overline{\mathscr{P}}_1{}^2 & \overline{\mathscr{P}}_2{}^2 \end{pmatrix}\right).$$

Equations (15.4)a follow from (15.7)a,b.

16. Local connections in $\mathscr{S}_3(O)$. [1]

DEFINITION (16.1). *Any set of constants* $\Xi_{\mathbf{a}}{}^{\mathbf{b}}{}_\lambda$ *which are transformed by* (5.7) [2] *according to*

$$(16.1) \qquad {}'\Xi_{\mathbf{a}'}{}^{\mathbf{b}}{}_\lambda = (\mathscr{C}_{\mathbf{i}}{}^{\mathbf{b}} \overset{-1}{\mathscr{C}}_{\mathbf{a}}{}^{\mathbf{j}})_O \Xi_{\mathbf{j}}{}^{\mathbf{i}}{}_\lambda + \mathscr{C}_{\mathbf{i}}{}^{\mathbf{b}}(O)(\partial_\lambda \overset{-1}{\mathscr{C}}_{\mathbf{a}}{}^{\mathbf{i}}(U))_O$$

will be referred to as a local connection in $\mathscr{S}_3(O)$.

THEOREM (16.1). *The set of constants*

$$(16.2)\text{a} \qquad {}^O\Pi_{\mathbf{a}}{}^{\mathbf{b}}{}_\lambda \overset{\text{def}}{=} (\partial_\lambda \mathscr{P}_{\mathbf{a}}{}^{\mathbf{b}}(U, O))_O$$

---

[1] The method displayed in this section is a special case of a more general theory developed in Bortolotti, E., Hlavatý, V., (1). See also Hlavatý, V., (9).

[2] We have here two transformations (5.7), one in $\mathscr{S}_3(O)$ with $\Phi = \Phi(O)$ and one in $\mathscr{S}_3(U)$ with $\Phi = \Phi(U)$. Due to the next to the last Note in section 5 we have to assume here $\epsilon_1 \epsilon_2 = 1$ because of requirement c) of Definition (15.1).

constitutes a local connection in $\mathscr{S}_3(O)$. *It is given by*

$$((^0\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda)) = \left(\begin{pmatrix} a_\lambda & b_\lambda & 0 & 0 \\ c_\lambda & d_\lambda & 0 & 0 \\ 0 & 0 & \bar{a}_\lambda & \bar{b}_\lambda \\ 0 & 0 & \bar{c}_\lambda & \bar{d}_\lambda \end{pmatrix}\right)_O \tag{16.2)b}$$

where $(a_\lambda)_O$, $(b_\lambda)_O$, $(c_\lambda)_O$, $(d_\lambda)_O$ *are the values at* O *of four arbitrary gradients.*

PROOF. We see from (15.5) that the coefficients $\mathscr{P}_{\mathrm{a}}{}^{\mathrm{b}}(U, O)$ are transformed by (5.7) according to

$$'\mathscr{P}_{\mathrm{a}}{}^{\mathrm{b}}(U, O) = \mathscr{C}_{\mathrm{i}}{}^{\mathrm{b}}(O)\overset{-1}{\mathscr{C}}_{\mathrm{a}}{}^{\mathrm{j}}(U)\mathscr{P}_{\mathrm{j}}{}^{\mathrm{i}}(U, O). \tag{16.3}$$

Remembering that $\mathscr{C}_{\mathrm{i}}{}^{\mathrm{b}}(O)$ are constant, we obtain from (16.3)

$$\begin{aligned} &\partial_\lambda{}'\mathscr{P}_{\mathrm{a}}{}^{\mathrm{b}}(U, O) \\ &= \mathscr{C}_{\mathrm{i}}{}^{\mathrm{b}}(O)[\mathscr{P}_{\mathrm{j}}{}^{\mathrm{i}}(U, O)\partial_\lambda\overset{-1}{\mathscr{C}}_{\mathrm{a}}{}^{\mathrm{j}}(U) + \overset{-1}{\mathscr{C}}_{\mathrm{a}}{}^{\mathrm{j}}(U)\partial_\lambda\mathscr{P}_{\mathrm{j}}{}^{\mathrm{i}}(U, O)]. \end{aligned} \tag{16.4}$$

Therefore, we obtain from (15.3)

$$\begin{aligned} &(\partial_\lambda{}'\mathscr{P}_{\mathrm{a}}{}^{\mathrm{b}}(U, O))_O \\ &= \mathscr{C}_{\mathrm{i}}{}^{\mathrm{b}}(O)(\partial_\lambda\overset{-1}{\mathscr{C}}_{\mathrm{a}}{}^{\mathrm{i}}(U))_O + (\mathscr{C}_{\mathrm{i}}{}^{\mathrm{b}}\overset{-1}{\mathscr{C}}_{\mathrm{a}}{}^{\mathrm{j}})_O(\partial_\lambda\mathscr{P}_{\mathrm{j}}{}^{\mathrm{i}}(U, O))_O. \end{aligned} \tag{16.5}$$

If we put

$$'^0\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda \overset{\mathrm{def}}{=} (\partial_\lambda{}'\mathscr{P}_{\mathrm{a}}{}^{\mathrm{b}}(U, O))_O,$$

we obtain the transformation rule (16.1)

$$'^0\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda = (\mathscr{C}_{\mathrm{i}}{}^{\mathrm{b}}\overset{-1}{\mathscr{C}}_{\mathrm{a}}{}^{\mathrm{j}})_O{}^0\Pi_{\mathrm{j}}{}^{\mathrm{i}}{}_\lambda + \mathscr{C}_{\mathrm{i}}{}^{\mathrm{b}}(O)(\partial_\lambda\overset{-1}{\mathscr{C}}_{\mathrm{a}}{}^{\mathrm{i}}(U))_O. \tag{16.6}$$

Moreover, we see from (16.2)a that $^0\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda$ may be looked upon as the values at O of sixteen gradients (which depend on transformations (5.7)). This fact together with (15.4)a leads to (16.2)b.

THEOREM (16.2). *The coefficients* $^0\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda$ *of the local connection have the following properties*:

a) *The expressions* $^0\Pi_{\mathrm{a}}{}^{\mathrm{a}}{}_\lambda$ *are invariant with respect to* (5.7)

$$'^0\Pi_{\mathrm{a}}{}^{\mathrm{a}}{}_\lambda = {}^0\Pi_{\mathrm{a}}{}^{\mathrm{a}}{}_\lambda. \tag{16.7}$$

b) *If we transform* $\mathscr{P}_{\mathfrak{a}}{}^{\mathfrak{b}}$ *according to the rule*

$$\hat{\mathscr{P}}_{\mathfrak{a}}{}^{\mathfrak{b}}(U, O) = \rho(U, O)\mathscr{P}_{\mathfrak{a}}{}^{\mathfrak{b}}(U, O) \tag{16.8)a}$$

$$\rho(O, O) = 1,^{1} \tag{16.8)b}$$

then

$$^{0}\hat{\Pi}_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda} = {}^{0}\Pi_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda} + r_{\lambda}(O)\delta^{\mathfrak{b}}_{\mathfrak{a}} \tag{16.9)a}$$

$$r_{\lambda}(O) \overset{\text{def}}{=} (\partial_{\lambda} \ln \rho(U, O))_{O}. \tag{16.9)b}$$

PROOF. We have from (5.7)

$$\mathscr{C}_{1}{}^{\mathfrak{a}}(O)\ (\partial_{\lambda}\overset{-1}{\mathscr{C}}_{\mathfrak{a}}{}^{1}(U))_{O} = 0. \tag{16.10}$$

This equation together with (16.6) leads to (16.7). Using (16.8) we obtain by virtue of (15.3) and (16.2)a

$$\begin{aligned} {}^{0}\hat{\Pi}_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda} &= (\partial_{\lambda}\rho(U, O)\mathscr{P}_{\mathfrak{a}}{}^{\mathfrak{b}}(U, O))_{O} \\ &= \rho(O, O)[r_{\lambda}(U, O)\mathscr{P}_{\mathfrak{a}}{}^{\mathfrak{b}}(U, O) + \partial_{\lambda}\mathscr{P}_{\mathfrak{a}}{}^{\mathfrak{b}}(U, O)]_{O} \\ &= {}^{0}\Pi_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda} + r_{\lambda}(O)\delta^{\mathfrak{b}}_{\mathfrak{a}}. \end{aligned}$$

In the next Theorem we assume that O and U are connected by a curve $x^{\nu} = x^{\nu}(t)$ of class C_2.

THEOREM (16.3). *Let* $\mathfrak{z}^{\mathfrak{a}}(U)$ *be an arbitrary spinor in* $\mathscr{S}_3(U)$ *of class* C_2. *Put*

$$\left(\frac{dx^{\lambda}}{dt}\,{}^{0}\mathscr{D}_{\lambda}\mathfrak{z}^{\mathfrak{a}}(U)\right)_{O} \overset{\text{def}}{=} \lim_{t\to 0} \frac{\mathfrak{z}^{\mathfrak{a}}(U|O) - \mathfrak{z}^{\mathfrak{a}}(O)}{t}. \tag{16.11)a}$$

Then

$$({}^{0}\mathscr{D}_{\lambda}\mathfrak{z}^{\mathfrak{b}}(U))_{O} = [\partial_{\lambda}\mathfrak{z}^{\mathfrak{b}}(U) + {}^{0}\Pi_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda}\,\mathfrak{z}^{\mathfrak{a}}(U)]_{O}. \tag{16.11)b}$$

Moreover, $\left(\frac{dx^{\lambda}}{dt}\,{}^{0}\mathscr{D}_{\lambda}\mathfrak{z}^{\mathfrak{b}}(U)\right)_{O}$ *is again a spinor. If, in particular,* $\mathfrak{z}^{\mathfrak{b}}(U)$ *is a physical spinor, so also is* $\left(\frac{dx^{\lambda}}{dt}\,{}^{0}\mathscr{D}_{\lambda}\mathfrak{z}^{\mathfrak{b}}(U)\right)_{O}$.

---

[1] $\hat{\mathscr{P}}_{\mathfrak{a}}{}^{\mathfrak{b}}$ and $\mathscr{P}_{\mathfrak{a}}{}^{\mathfrak{b}}$ represent the same projectivity because the components are homogeneous. Therefore, we have to assume (16.8)b to obtain

$$\hat{\mathscr{P}}_{\mathfrak{a}}{}^{\mathfrak{b}}(O, O) = \delta^{\mathfrak{b}}_{\mathfrak{a}}$$

in accordance with requirement a) of Definition (15.1).

PROOF. The spinor $z^{\mathrm{b}}(U)$ being of class C_2 we have by virtue of (15.5), (15.4)b and (16.2)a

$$z^{\mathrm{b}}(U|O) = (\delta^{\mathrm{b}}_{\mathrm{a}} + x^{\lambda}\,{}^{0}\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda} + \dots)(z^{\mathrm{a}}(O) + x^{\mu}(\partial_{\mu} z^{\mathrm{a}}(U))_O + \dots) \tag{16.12)a}$$
$$= z^{\mathrm{b}}(O) + x^{\lambda}[(\partial_{\lambda} z^{\mathrm{b}}(U))_O + {}^{0}\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda} z^{\mathrm{a}}(O)] + \dots .$$

If we substitute into this equation

$$x^{\lambda} = t\frac{dx^{\lambda}}{dt} + \dots, \tag{16.12)b}$$

we easily obtain (16.11)a,b. The proof of the remaining part of our Theorem follows at once from (16.6) and (16.2)b.

NOTE I. According to (16.11)b ${}^{0}\mathscr{D}_{\lambda}$ may be looked upon as the symbol of the covariant derivative in $\mathscr{S}_3(O)$ with respect to the local connection ${}^{0}\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda}$.

NOTE II. We see from previous results that the spinor $z^{\mathrm{b}}(U)$ is projected into $z^{\mathrm{b}}(O)$ if and only if

$$\left[\frac{dx^{\lambda}}{dt}\,{}^{0}\mathscr{D}_{\lambda} z^{\mathrm{b}}(U) - z^{\mathrm{b}}(U)a(U)\right]_O = 0 \tag{16.13}$$

where $a(U)$ is a suitable function of position. This fact shows clearly the geometrical significance of the operator $\dfrac{dx^{\lambda}}{dt}\,{}^{0}\mathscr{D}_{\lambda}$. The form of (16.13) remains unchanged if we perform the transformation (16.8) and

$${}^{*}z^{\mathrm{b}}(U) = f(U)z^{\mathrm{b}}(U),\ f \neq 0. \tag{16.14}$$

THEOREM (16.4). *With every local connection* ${}^{0}\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda}$ *there is associated a local connection* $\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda}$ *invariant with respect to* (16.8)

$$\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda} \overset{\mathrm{def}}{=} {}^{0}\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda} - \tfrac{1}{4}\,{}^{0}\Pi_{\mathrm{i}}{}^{\mathrm{i}}{}_{\lambda}\delta^{\mathrm{b}}_{\mathrm{a}}. \tag{16.15)a}$$

This connection satisfies the condition

$$\Pi_{\mathrm{i}}{}^{\mathrm{i}}{}_{\lambda} = 0 \tag{16.16}$$

invariant with respect to (5.7). *The constants* $\Pi_{\mathrm{b}}{}^{\mathrm{a}}{}_{\lambda}$ *may be expressed* *as follows:*

$$(16.15)\text{b} \qquad ((\Pi_{\mathrm{b}}{}^{\mathrm{a}}{}_\lambda)) = \left(\!\!\left(\begin{matrix} a_\lambda & b_\lambda & 0 & 0 \\ c_\lambda & d_\lambda & 0 & 0 \\ 0 & 0 & \bar{a}_\lambda & \bar{b}_\lambda \\ 0 & 0 & \bar{c}_\lambda & \bar{d}_\lambda \end{matrix}\right)\!\!\right)$$ [1]

where a_λ, b_λ, c_λ, d_λ are the values at O of four gradients, and

$$(16.15)\text{c} \qquad R(a_\lambda) + R(d_\lambda) = 0.$$ [2]

PROOF. The equations (16.9)a yield

$$(16.17) \qquad r_\lambda(O) = \tfrac{1}{4}({}^0\hat{\Pi}_{\mathrm{i}}{}^{\mathrm{i}}{}_\lambda - {}^0\Pi_{\mathrm{i}}{}^{\mathrm{i}}{}_\lambda).$$

Substituting this value in (16.9)a one obtains

$$(16.18) \qquad \begin{aligned} \hat{\Pi}_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda &\overset{\text{def}}{=} {}^0\hat{\Pi}_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda - \tfrac{1}{4}{}^0\hat{\Pi}_{\mathrm{i}}{}^{\mathrm{i}}{}_\lambda \delta^{\mathrm{b}}_{\mathrm{a}} \\ &= {}^0\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda - \tfrac{1}{4}{}^0\Pi_{\mathrm{i}}{}^{\mathrm{i}}{}_\lambda \delta^{\mathrm{b}}_{\mathrm{a}} = \Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda. \end{aligned}$$

This equation shows the invariance of $\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda$ with respect to (16.8). The equation (16.7) shows that the constants $\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda$ defined by (16.15)a constitute a connection because ${}^0\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda$ constitute a connection. The equation (16.16) is an immediate consequence of (16.15)a. Its invariance with respect to (5.7) follows from (16.10). The equations (16.15)b,c follow from (16.2)b and (16.16), because

$$a_\lambda + d_\lambda + \bar{a}_\lambda + \bar{d}_\lambda = 2(R(a_\lambda) + R(d_\lambda)).$$

17. Continuation. Among all invariant local connections $\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda$ there is one which is *uniquely determined by the unified tensor field* $g_{\lambda\mu}$. In order to find it we need some auxiliary Lemmas about the nonholonomic components $\{{}_j{}^i{}_k\}$ of the Christoffel symbols given by II (13.1). In the following Lemma we shall consider the functions

$$(17.1)\text{a} \qquad \Omega_B{}^A{}_\lambda \overset{\text{def}}{=} \delta^A_i \, \delta^j_B \{{}_j{}^i{}_k\} \overset{k}{a}_\lambda = \delta^A_i \, \delta^j_B \overset{i}{a}_\alpha \nabla_\lambda \underset{j}{a}{}^\alpha.$$

LEMMA (17.1). *The functions $\Omega_B{}^A{}_\lambda$ satisfy the following relations:*

$$(17.1)\text{b} \quad \Omega_5{}^5{}_\lambda = \Omega_6{}^5{}_\lambda = \Omega_5{}^6{}_\lambda = \Omega_6{}^6{}_\lambda = \Omega_2{}^1{}_\lambda = \Omega_1{}^2{}_\lambda = \Omega_3{}^4{}_\lambda = \Omega_4{}^3{}_\lambda = 0$$

---

[1] The vectors a_λ, d_λ in (16.15)b are not the same as the vectors a_λ, d_λ in (16.2)b. Both of these sets of vectors are connected by two relations which may easily be derived from (16.15)a.

[2] $R(n)$ is the real part of a complex number n.

$$(17.1)\text{c} \qquad \Omega_4{}^1{}_\lambda = -\Omega_2{}^3{}_\lambda,\ \Omega_3{}^2{}_\lambda = -\Omega_1{}^4{}_\lambda,\ \Omega_4{}^2{}_\lambda = -\Omega_1{}^3{}_\lambda,$$

$$\Omega_3{}^1{}_\lambda = -\Omega_2{}^4{}_\lambda,\ \Omega_1{}^1{}_\lambda = -\Omega_2{}^2{}_\lambda,\ \Omega_3{}^3{}_\lambda = -\Omega_4{}^4{}_\lambda,$$

$$(17.1)\text{d} \qquad \Omega_f{}^2{}_\lambda = \bar{\Omega}_f{}^1{}_\lambda,\ \Omega_1{}^f{}_\lambda = \bar{\Omega}_2{}^f{}_\lambda,\ \Omega_1{}^1{}_\lambda = \bar{\Omega}_2{}^2{}_\lambda,$$

$$(f = 3, 4),$$

$$(17.1)\text{e} \qquad \Omega_A{}^A{}_\lambda = 0.$$

The equations (17.1)b,c,d,e *are invariant with respect to the transformations* (1.11).

PROOF. The first four terms in (17.1)b are equal to zero because of (17.1)a. The vector $\underset{2}{a}^\nu$ is a null vector, and therefore, by virtue of I (8.16)

$$\Omega_2{}^1{}_\lambda = \overset{1}{a}_\alpha \nabla_\lambda \underset{2}{a}^\alpha = \underset{2}{a}_\alpha \nabla_\lambda \underset{2}{a}^\alpha = \tfrac{1}{2} \nabla_\lambda h_{\alpha\beta} \underset{2}{a}^\alpha \underset{2}{a}^\beta = 0.$$

The remaining equations (17.1)b may be proved in a similar way. Again using I (8.16) we obtain, according to I (8.1)b,

$$\Omega_4{}^1{}_\lambda = \overset{1}{a}_\alpha \nabla_\lambda \underset{4}{a}^\alpha = \underset{2}{a}_\alpha \nabla_\lambda \overset{3}{a}^\alpha = - \overset{3}{a}_\alpha \nabla_\lambda \underset{2}{a}^\alpha = - \Omega_2{}^3{}_\lambda.$$

The remaining equations (17.1)c may be proved in the same manner. The vectors $\underset{1}{a}^\nu$ and $\underset{2}{a}^\nu$ are complex conjugates (*Cf*. Theorem I (7.4)). Therefore, according to I (8.16), the vectors $\overset{1}{a}_\lambda$ and $\overset{2}{a}_\lambda$ are also complex conjugates. Thus,

$$\Omega_f{}^2{}_\lambda = \overset{2}{a}_\alpha \nabla_\lambda \underset{f}{a}^\alpha = \overset{1}{\bar{a}}_\alpha \nabla_\lambda \underset{f}{a}^\alpha = \bar{\Omega}_f{}^1{}_\lambda \ (f = 3, 4).$$

In a similar way we obtain the remaining equations (17.1)d. The equations (17.1)e follow from the last two equations (17.1)c and the first and fourth equations (17.1)b. The invariance of (17.1)b-e with respect to (1.11) follows easily from I (8.1)b, (1.11) and (17.1)a.

REMARK. Note that

$$(17.2) \qquad {}'\Omega_j{}^i{}_\lambda = C_b{}^i \overset{-1}{C}_j{}^a \Omega_a{}^b{}_\lambda + C_b{}^i \partial_\lambda \overset{-1}{C}_j{}^b.$$

In subsequent considerations $\mathscr{D}_\lambda$ will be the symbol of the covariant derivative with respect to the connections $\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda}$ and $\Omega_{B}{}^{A}{}_{\lambda}$. Thus, for instance, if $T_{B}{}^{\mathrm{a}}(U)$ is a $(C, \mathscr{C})$-tensor,

$$(\mathscr{D}_{\lambda}T_{\mathrm{B}}{}^{\mathrm{a}}(U))_{O} = [\partial_{\lambda}T_{B}{}^{\mathrm{a}} + \Pi_{\mathrm{b}}{}^{\mathrm{a}}{}_{\lambda}T_{B}{}^{\mathrm{b}} - \Omega_{B}{}^{A}{}_{\lambda}T_{A}{}^{\mathrm{a}}]_{O}.$$

An easy inspection shows that $(dx^{\lambda}\mathscr{D}_{\lambda}T_{B}{}^{\mathrm{a}}(U))_{O}$ is again a $(C, \mathscr{C})$-tensor and of the same tensorial character as $T_{B}{}^{\mathrm{a}}(O)$.

THEOREM (17.1). *Among all possible invariant local connections* $\Pi_{\mathrm{b}}{}^{\mathrm{a}}{}_{\lambda}$ *there is one uniquely determined by* $g_{\lambda\mu}$. *It is the invariant local connection for which*

(17.3)a $$(\mathscr{D}_{\lambda}\delta^{\mathrm{ab}}_{A})_{O} = 0.^{1}$$

If this equation is satisfied, then

(17.3)b $$(\mathscr{D}_{\lambda}\delta^{A}_{\mathrm{ab}})_{O} = 0.$$

This connection is given by the equations

(17.4)a $$((\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda})) = \left(\begin{pmatrix} a_{\lambda} & b_{\lambda} & 0 & 0 \\ c_{\lambda} & -a_{\lambda} & 0 & 0 \\ 0 & 0 & \bar{a}_{\lambda} & \bar{b}_{\lambda} \\ 0 & 0 & \bar{c}_{\lambda} & -\bar{a}_{\lambda} \end{pmatrix}\right)$$

at O. The constants a_{λ}, b_{λ}, c_{λ} *are given by the equations*

(17.4)b $$2a_{\lambda} = \overset{1}{a}_{\alpha}\nabla_{\lambda}\underset{1}{a}{}^{\alpha} + \overset{3}{a}_{\alpha}\nabla_{\lambda}\underset{3}{a}{}^{\alpha},$$

(17.4)c $$b_{\lambda} = \overset{1}{a}_{\alpha}\nabla_{\lambda}\underset{4}{a}{}^{\alpha},$$

(17.4)d $$c_{\lambda} = \overset{4}{a}_{\alpha}\nabla_{\lambda}\underset{1}{a}{}^{\alpha},$$

at O.

PROOF. The condition (17.3)a is equivalent to

(17.5)a $$\Omega_{A}{}^{B}{}_{\lambda}(O)\delta^{\mathrm{ab}}_{B} = \Pi_{\mathrm{j}}{}^{\mathrm{a}}{}_{\lambda}\delta^{\mathrm{jb}}_{A} + \Pi_{\mathrm{j}}{}^{\mathrm{b}}{}_{\lambda}\delta^{\mathrm{aj}}_{A}.$$

A straightforward calculation shows that (17.5)a is equivalent to

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[1] For δ^{ab}_{A} and δ^{A}_{ab} see Theorems (2.3)a,b.

$$(17.5)\text{b} \qquad ((\Omega_A{}^B{}_\lambda(O)))=$$

$$\left(\!\!\left(\begin{matrix}
\Pi_1{}^1{}_\lambda+\Pi_2{}^2{}_\lambda & 0 & -\Pi_3{}^4{}_\lambda & \Pi_2{}^1{}_\lambda & \Pi_2{}^4{}_\lambda & \Pi_3{}^1{}_\lambda \\
0 & \Pi_2{}^2{}_\lambda+\Pi_3{}^3{}_\lambda & -\Pi_1{}^2{}_\lambda & \Pi_4{}^3{}_\lambda & -\Pi_1{}^3{}_\lambda & -\Pi_4{}^2{}_\lambda \\
-\Pi_4{}^3{}_\lambda & -\Pi_2{}^1{}_\lambda & \Pi_3{}^3{}_\lambda+\Pi_1{}^1{}_\lambda & 0 & -\Pi_2{}^3{}_\lambda & \Pi_4{}^1{}_\lambda \\
\Pi_1{}^2{}_\lambda & \Pi_3{}^4{}_\lambda & 0 & \Pi_2{}^2{}_\lambda+\Pi_4{}^4{}_\lambda & -\Pi_1{}^4{}_\lambda & \Pi_3{}^2{}_\lambda \\
\Pi_4{}^2{}_\lambda & -\Pi_3{}^1{}_\lambda & -\Pi_3{}^2{}_\lambda & -\Pi_4{}^1{}_\lambda & \Pi_1{}^1{}_\lambda+\Pi_2{}^2{}_\lambda & 0 \\
\Pi_1{}^3{}_\lambda & -\Pi_2{}^4{}_\lambda & \Pi_1{}^4{}_\lambda & \Pi_2{}^3{}_\lambda & 0 & \Pi_3{}^3{}_\lambda+\Pi_4{}^4{}_\lambda
\end{matrix}\right)\!\!\right).$$

Substituting into the left-hand side from (17.1) and into the right-hand side from (16.15)b, one obtains (17.4)a,b,c,d and the complex-conjugate equations to (17.4)b,c,d. The remaining equations (17.5)b reduce to an identity 0=0. The equation (17.3)b follows from (17.3)a and from (2.7). The vectors $\underset{j}{a}{}^\nu$, $\overset{i}{a}_\lambda$ involved in (17.4) are determined by $g_{\lambda\mu}$ up to the transformations (1.11). Thus,

$$(17.6) \qquad \left.\begin{aligned} 'a_\lambda &= a_\lambda + \tfrac{1}{2}\partial_\lambda(\psi + i\varphi) \\ 'b_\lambda &= \epsilon' e^{-(\psi+i\varphi)} b_\lambda \\ 'c_\lambda &= \epsilon' e^{\psi+i\varphi} c_\lambda \end{aligned}\right\} \text{ at } O.$$

Using (17.4)a and (5.7) we see that (17.6) is only another form of the transformation rule

$$(17.7) \qquad '\Pi_{\mathfrak{b}}{}^{\mathfrak{a}}{}_\lambda = [\mathscr{C}_{\mathfrak{j}}{}^{\mathfrak{a}} \overset{-1}{\mathscr{C}}_{\mathfrak{b}}{}^{\mathfrak{j}} \, \Pi_{\mathfrak{j}}{}^{\mathfrak{i}}{}_\lambda + \mathscr{C}_{\mathfrak{j}}{}^{\mathfrak{a}} \partial_\lambda \overset{-1}{\mathscr{C}}_{\mathfrak{b}}{}^{\mathfrak{j}}]_O$$

which characterizes the set $\Pi_{\mathfrak{b}}{}^{\mathfrak{a}}{}_\lambda$ as a local connection. Hence, the connection (17.4) is uniquely determined by $g_{\lambda\mu}$.

NOTE. The equations (17.5) show that the requirements (17.1)b are necessary for our construction of $\Pi_{\mathfrak{b}}{}^{\mathfrak{a}}{}_\lambda$. In fact, the set (17.1)b is a subset of (17.5)a. This limits considerably the choice of the connection Π. Thus, for instance, if $\Gamma_j{}^i{}_k$ are nonholonomic components of the unified connection II (3.1), and if we put

$$\Omega_B{}^A{}_\lambda \overset{\text{def}}{=} \delta^A_i \delta^j_B \Gamma_j{}^i{}_k \overset{k}{a}_\lambda,$$

then

$$\Omega_2{}^1{}_\lambda = (S_{\mu\lambda}{}^\nu + U^\nu{}_{\lambda\mu}) \underset{2}{a}{}^\mu \overset{1}{a}_\nu$$

$$= S_{2\lambda}{}^1 + U^1{}_{\lambda 2} \neq 0.$$

This is the reason we did not use this connection for the construction of $\Pi_{\mathfrak{b}}{}^{\mathfrak{a}}{}_{\lambda}$.

18. Spinor space $\mathscr{S}_3$ and its connection $\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda}$. The rulings

$$\underset{1}{a}^{[\nu}\underset{3}{a}^{\lambda]},\ \underset{2}{a}^{[\nu}\underset{4}{a}^{\lambda]},\ \underset{2}{a}^{[\nu}\underset{3}{a}^{\lambda]},\ \underset{1}{a}^{[\nu}\underset{4}{a}^{\lambda]} \tag{18.1}$$

of the quadric $H(O)$ in $T_4(O)$ are mapped into the focal spinors (4.4) in $\mathscr{S}_3(O)$. The rulings (18.1) of the quadric $H(U)$ in $T_4(U)$ are mapped into the focal spinors ${}^1\underset{i}{a}^{\mathfrak{a}}(U)$, ${}^2\underset{i}{a}^{\mathfrak{a}}(U)$ whose homogeneous components are given again by (4.4).

Denote by $z^{\mathfrak{a}}(U\|O)$ a spinor in $\mathscr{S}_3(O)$ whose components with respect to $\underset{j}{a}^{\mathfrak{j}}(O)$ are the same as the components of the spinor $z^{\mathfrak{a}}(U)$ in $\mathscr{S}_3(U)$ with respect to $\underset{j}{a}^{\mathfrak{j}}(U)$. As $z^{\mathfrak{a}}(U)$ moves along $\mathscr{S}_3(U)$ the spinor $z^{\mathfrak{a}}(U\|O)$ describes a space superimposed onto $\mathscr{S}_3(O)$. We shall denote this space by $\mathscr{S}_3(U\|O)$. It is clear that $\mathscr{S}_3(U)$ and $\mathscr{S}_3(U\|O)$ are projectively related.

DEFINITION (18.1). *The set of all spaces $\mathscr{S}_3(U\|O)$ superimposed onto $\mathscr{S}_3(O)$ as U moves throughout space-time will be referred to as the spinor space $\mathscr{S}_3$ adjoint to the space-time.*

NOTE I. The axes ${}^1\mathscr{Q}^{\mathfrak{ab}}(U\|O)$ and ${}^2\mathscr{Q}^{\mathfrak{ab}}(U\|O)$ coincide with ${}^1\mathscr{Q}^{\mathfrak{ab}}(O)$ and ${}^2\mathscr{Q}^{\mathfrak{ab}}(O)$ for any U.

NOTE II. The spinor algebra of $\mathscr{S}_3(U\|O)$ is the same as the spinor algebra of the local spinor space $\mathscr{S}_3(O)$ developed in the first part of this chapter. The only difference is purely formal. The objects of $\mathscr{S}_3(U\|O)$ have the argument $U\|O$ instead of O. *The spinor algebra of $\mathscr{S}_3$ is by definition the spinor algebra of each individual space* $\mathscr{S}_3(U\|O)$. In algebraic considerations of $\mathscr{S}_3$ an object $X(\underset{1}{U}\|O)$ of $\mathscr{S}_3(\underset{1}{U}\|O)$ is *in general* different from the corresponding object $X(\underset{2}{U}\|O)$ of $\mathscr{S}_3(\underset{2}{U}\|O)$ for $\underset{1}{U} \neq \underset{2}{U}$.[1]

Spinor analysis of $\mathscr{S}_3$ requires a spinor connection, say $\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda}$, which is characterized by the usual law of transformation

$$'\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda} = \mathscr{C}_{\mathfrak{j}}{}^{\mathfrak{b}}\overset{-1}{\mathscr{C}}_{\mathfrak{a}}{}^{\mathfrak{i}}\Lambda_{\mathfrak{i}}{}^{\mathfrak{j}}{}_{\lambda} + \mathscr{C}_{\mathfrak{j}}{}^{\mathfrak{b}}\,\partial_{\lambda}\overset{-1}{\mathscr{C}}_{\mathfrak{a}}{}^{\mathfrak{j}} \tag{18.2}$$

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[1] An exception to this rule is exhibited in the previous Note I.

where, of course, the coefficients are given by (5.7) with $\varphi = \varphi(U)$ and $\psi = \psi(U)$. Hence, we cannot use any one of the local connections exhibited in previous sections here.

In the next Theorem, where we shall find a connection $\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_\lambda$ with the transformation law (18.2), $\mathscr{L}_\mu$ will be the symbol of the covariant derivative with respect to the following three connections:

a) the spinor connection $\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_\lambda$ we are looking for,

b) the connection $\Omega_B{}^A{}_\lambda$ defined by (17.1)a,

c) the connection given by the Christoffel symbols $\{{}_\lambda{}^\nu{}_\mu\}$.

Thus, if we have, for instance, a tensor $T_B{}^{\mathfrak{a}}{}_\mu$ with three different types of indices, we put

$$(18.3) \qquad \mathscr{L}_\lambda T_B{}^{\mathfrak{a}}{}_\mu \overset{\text{def}}{=} \partial_\lambda T_B{}^{\mathfrak{a}}{}_\mu + \Lambda_{\mathfrak{b}}{}^{\mathfrak{a}}{}_\lambda T_B{}^{\mathfrak{b}}{}_\mu - \Omega_B{}^A{}_\lambda T_A{}^{\mathfrak{a}}{}_\mu - \{{}_\mu{}^\nu{}_\lambda\} T_B{}^{\mathfrak{a}}{}_\nu.$$

An easy inspection shows that for any dx^λ the components $dx^\lambda \mathscr{L}_\lambda T_B{}^{\mathfrak{a}}{}_\mu$ define another tensor of the same tensorial character as $T_B{}^{\mathfrak{a}}{}_\mu$.

THEOREM (18.1). *There is one connection* $\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_\lambda$ *uniquely determined by* $g_{\lambda\mu}$. *It is defined by*

$$(18.4)\text{a} \qquad \mathscr{L}_\lambda \delta^{\mathfrak{ab}}_A = 0.$$

If this equation is satisfied, then

$$(18.4)\text{b} \qquad \mathscr{L}_\lambda \delta^A_{\mathfrak{ab}} = 0$$

and

$$(18.4)\text{c} \qquad \mathscr{L}_\lambda E_{\mu\mathfrak{a}}{}^{\mathfrak{b}} = 0.$$ [1]

This connection is given by

$$(18.5) \qquad ((\Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_\lambda)) = \left(\begin{pmatrix} a_\lambda & b_\lambda & 0 & 0 \\ c_\lambda & -a_\lambda & 0 & 0 \\ 0 & 0 & \bar{a}_\lambda & \bar{b}_\lambda \\ 0 & 0 & \bar{c}_\lambda & -\bar{a}_\lambda \end{pmatrix}\right)$$

where the vector fields a_λ, b_λ, c_λ *are given by* (17.4)b,c,d *for all* U*'s.*

The proof of this Theorem, except for (18.4)c, follows the pattern of the proof of Theorem (17.1). The equation (18.4)c follows by straightforward computation based on (1.14), (18.5) and (17.4)b,c,d.

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[1] For $E_{\mu\mathfrak{a}}{}^{\mathfrak{b}}$ see (1.14).

Thus, for instance, according to (1.14) and (18.5)

$$(18.6)\qquad \mathscr{L}_\lambda E_{\mu 3}{}^{1} = \partial_\lambda \overset{1}{a}_\mu - \{\mu{}^\nu{}\lambda\}\overset{1}{a}_\nu + \overset{1}{a}_\mu(a_\lambda - \bar{a}_\lambda) - \overset{3}{\bar{c}}_\lambda a_\mu + \overset{4}{b}_\lambda a_\mu .$$

On the other hand, we get by virtue of (17.4)b,c,d

$$\begin{aligned}
\overset{1}{a}_\mu(a_\lambda - \bar{a}_\lambda) &= \tfrac{1}{2}\overset{1}{a}_\mu(\overset{1}{a}_\alpha \nabla_\lambda \underset{1}{a}^\alpha - \overset{2}{a}_\alpha \nabla_\lambda \underset{2}{a}^\alpha)\\
&= \tfrac{1}{2}\overset{1}{a}_\mu(\overset{1}{a}_\alpha \nabla_\lambda \underset{1}{a}^\alpha + \overset{1}{a}_\alpha \nabla_\lambda \underset{1}{a}^\alpha) = \overset{1}{a}_\mu \overset{1}{a}_\alpha \nabla_\lambda \underset{1}{a}^\alpha\\
&= -\,\overset{1}{a}_\mu \underset{1}{a}^\alpha \nabla_\lambda \overset{1}{a}_\alpha\\
-\,\overset{3}{\bar{c}}_\lambda a_\mu &= -\,\overset{3}{a}_\mu \overset{4}{a}_\alpha \nabla_\lambda \underset{2}{a}^\alpha = -\,\overset{3}{a}_\mu \overset{4}{a}_\alpha \nabla_\lambda \overset{1}{a}^\alpha\\
&= -\,\overset{3}{a}_\mu \underset{3}{a}^\alpha \nabla_\lambda \overset{1}{a}_\alpha\\
\overset{4}{b}_\lambda a_\mu &= \overset{4}{a}_\mu \overset{1}{a}_\alpha \nabla_\lambda \underset{4}{a}^\alpha = -\,\overset{4}{a}_\mu \underset{4}{a}^\alpha \nabla_\lambda \overset{1}{a}_\alpha .
\end{aligned}$$

Substituting these values into (18.6), we obtain according to I (8.1)a

$$\begin{aligned}
\mathscr{L}_\lambda E_{\mu 3}{}^{1} &= \nabla_\lambda \overset{1}{a}_\mu - \overset{1}{a}_\mu \underset{1}{a}^\alpha \nabla_\lambda \overset{1}{a}_\alpha - \overset{3}{a}_\mu \underset{3}{a}^\alpha \nabla_\lambda \overset{1}{a}_\alpha - \overset{4}{a}_\mu \underset{4}{a}^\alpha \nabla_\lambda \overset{1}{a}_\alpha\\
&= \overset{2}{a}_\mu \underset{2}{a}^\alpha \nabla_\lambda \overset{1}{a}_\alpha = 0.
\end{aligned}$$

The relationship between the connections $\Pi_{\mathrm{b}}{}^{\mathrm{a}}{}_\lambda$ and $\Lambda_{\mathrm{b}}{}^{\mathrm{a}}{}_\lambda$ is described in the following

THEOREM (18.2). *The connections* $\Pi_{\mathrm{b}}{}^{\mathrm{a}}{}_\lambda$ *and* $\Lambda_{\mathrm{b}}{}^{\mathrm{a}}{}_\lambda$ *coincide for* $U = O$

$$(18.7)\qquad \Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda = (\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda)_O .$$

The proof follows easily from (18.5), (17.4)a and (17.4)b,c,d. The latter equations are valid for both connections at O.

The equations (1.17) and (2.8)a,b, *i.e.*,

$$(18.8)\text{a}\qquad v_{\mathrm{a}}{}^{\mathrm{b}} = E_{\mu\mathrm{a}}{}^{\mathrm{b}} v^\mu$$

$$(18.8)\text{b}\qquad \epsilon_{\mathfrak{v}}{}^{\mathrm{ab}} = \sqrt{2}\,\delta^{\mathrm{ab}}_{A} v^A$$

$$(18.8)\text{c}\qquad \sqrt{2}\,v^A = \delta^{A}_{\mathrm{ab}}\,\epsilon_{\mathfrak{v}}{}^{\mathrm{ab}}$$

hold also in $\mathscr{S}_3$ where the quantities involved depend on U.

THEOREM (18.3). *We have*

$$\mathscr{L}_\lambda \mathfrak{v}_{\mathrm{a}}{}^{\mathrm{b}} = E_{\mu\mathrm{a}}{}^{\mathrm{b}}\mathscr{L}_\lambda v^\mu \tag{18.9)a}$$

$$\mathscr{L}_\lambda {}^{\epsilon}\mathfrak{v}^{\mathrm{ab}} = \sqrt{2}\,\delta^{\mathrm{ab}}_{A}\mathscr{L}_\lambda v^A \tag{18.9)b}$$

$$\sqrt{2}\,\mathscr{L}_\lambda v^A = \delta^{A}_{\mathrm{ab}}\mathscr{L}_\lambda {}^{\epsilon}\mathfrak{v}^{\mathrm{ab}}. \tag{18.9)c}$$

The proof follows from (18.4)a,b,c and (18.8)a,b,c.

NOTE. These equations clearly constitute the link between the spinor algebra and spinor analysis of $\mathscr{S}_3$.

NOTE. It is worth while to compare previous results with those which one can obtain in the gravitational theory.

1) The construction of $\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda$ by means of (18.5) requires $h_{\lambda\mu}$ and $\underset{j}{a}^\nu$, *i.e.*, the unified tensor field $g_{\lambda\mu}$:

$$\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda = \Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda(g..). \tag{18.10)a}$$

The basic vectors are determined by $g_{\lambda\mu}$ up to the transformations (1.11)a,b,c,d; the coefficients $'\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda$ defined by (18.2) are functions of $g_{\lambda\mu}$ and of *two additional parameters* φ, ψ

$$'\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda = {}'\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_\lambda(g.., \varphi, \psi). \tag{18.10)b}$$

However, these additional parameters φ, ψ are eliminated in tensorial equations involving the covariant derivative $\mathscr{L}_\lambda$. In fact, if, for instance, $\mathfrak{v}^{\mathrm{a}}$ is a generic spinor, and if we put

$$\mathscr{T}_j{}^{\mathrm{a}} \overset{\mathrm{def}}{=} \underset{j}{a}^\lambda \mathscr{L}_\lambda \mathfrak{v}^{\mathrm{a}},$$

we have

$$'\mathscr{T}_j{}^{\mathrm{a}}(g.., \varphi, \psi) = \overset{-1}{C}_j{}^k(\varphi, \psi)\mathscr{C}_{\mathrm{b}}{}^{\mathrm{a}}(\varphi, \psi)\mathscr{T}_k{}^{\mathrm{b}}(g..), \tag{18.11}$$

and this equation shows that *tensorial* equations involving $\mathscr{T}_j{}^{\mathrm{a}}$ will not contain the additional parameters φ, ψ.

2) In the gravitational theory we know only the tensor $h_{\lambda\mu}$. However, we may choose four vectors $\underset{j}{a}^\nu$ satisfying the conditions appearing in Theorem I (6.3). This choice requires *four additional parameters* besides $h_{\lambda\mu}$, for instance, the Plücker coordinates

$p^{\lambda\nu} = p^{[\lambda\nu]}$ of the line in P_3 joining $\underset{1}{a^\nu}$ and $\underset{2}{a^\nu}$.[1] Thus, we have according to (18.5) and (17.4)

$$\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda} = \Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda}(h_{..}, p^{..}) \tag{18.12)a}$$

instead of (18.10)a. If we start with another line

$$'p^{\lambda\nu} = A_\alpha{}^\lambda A_\beta{}^\nu p^{\alpha\beta}, \tag{18.13}$$

then we again obtain (1.11)a,b. This time the coefficients C are functions of the A's and are not given by (1.11)c,d. Finally, we obtain (18.2) where the coefficients $\mathscr{C}$ are functions of the A's and are not given by (5.7)a,b. Therefore, we obtain instead of (18.10)b and (18.11)

$$'\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda} = {}'\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda}(h_{..}, p^{..}, A{:}) \tag{18.12)b}$$

$$'\mathscr{T}_j{}^{\mathrm{a}}(h_{..}, p^{..}, A{:}) = \overset{-1}{C}{}_j{}^k(A{:})\mathscr{C}_{\mathrm{b}}{}^{\mathrm{a}}(A{:})\mathscr{T}_k{}^{\mathrm{b}}(h_{..}, p^{..}). \tag{18.14}$$

Hence, the influence of the auxiliary parameters $A_\lambda{}^\nu$ will be eliminated in tensorial equations. However, *in contradistinction to the unified field theory, the connection $\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda}$ in the gravitational theory depends on four additional parameters not connected with the given quantities, (i.e., not connected with $h_{\lambda\mu}$).*

19. Curvature tensor of $\mathscr{S}_3$. In subsequent considerations we shall need the curvature tensor $H_{\omega\mu\lambda}{}^\nu$ of $\{{}_\lambda{}^\nu{}_\mu\}$, defined by IV (3.1), and its intermediate components

$$\begin{aligned} H_{\omega\mu A}{}^B &= \delta_j^B \delta_A^i H_{\omega\mu\lambda}{}^\nu \underset{i}{a^\lambda} \overset{j}{a_\nu} \\ &= 2\delta_j^B \delta_A^i (\partial_{[\mu} \Omega_{|i|}{}^j{}_{\omega]} + \Omega_k{}^j{}_{[\mu} \Omega_{|i|}{}^k{}_{\omega]}). \end{aligned} \tag{19.1)a}$$

These components are involved in identities similar to IV (3.3)

$$\begin{aligned} 2\mathscr{L}_{[\omega}\mathscr{L}_{\mu]} v^A &= -H_{\omega\mu B}{}^A\, v^B \\ 2\mathscr{L}_{[\omega}\mathscr{L}_{\mu]} w_B &= H_{\omega\mu B}{}^A\, w_A. \end{aligned} \tag{19.2}$$

LEMMA (19.1). *The components $H_{\omega\mu A}{}^B$ satisfy the following set of*

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[1] The six homogeneous coordinates $p^{\lambda\nu}$ satisfy a quadratic equation and, therefore, represent only four parameters.

conditions:

(19.1)b $$H_{\omega\mu 5}{}^{5} = H_{\omega\mu 6}{}^{5} = H_{\omega\mu 5}{}^{6} = H_{\omega\mu 6}{}^{6} =$$
$$H_{\omega\mu 2}{}^{1} = H_{\omega\mu 1}{}^{2} = H_{\omega\mu 3}{}^{4} = H_{\omega\mu 4}{}^{3} = 0,$$

(19.1)c $$H_{\omega\mu 4}{}^{1} = -H_{\omega\mu 2}{}^{3},\quad H_{\omega\mu 3}{}^{2} = -H_{\omega\mu 1}{}^{4},\quad H_{\omega\mu 4}{}^{2} = -H_{\omega\mu 1}{}^{3},$$
$$H_{\omega\mu 3}{}^{1} = -H_{\omega\mu 2}{}^{4},\quad H_{\omega\mu 1}{}^{1} = -H_{\omega\mu 2}{}^{2},\quad H_{\omega\mu 3}{}^{3} = -H_{\omega\mu 4}{}^{4},$$

(19.1)d $$H_{\omega\mu f}{}^{2} = \bar{H}_{\omega\mu f}{}^{1},\quad H_{\omega\mu 1}{}^{f} = \bar{H}_{\omega\mu 2}{}^{f},\quad H_{\omega\mu 1}{}^{1} = \bar{H}_{\omega\mu 2}{}^{2},$$
$$(f = 3, 4),$$

(19.1)e $$H_{\omega\mu A}{}^{A} = 0.$$

The equations (19.1)b,c,d,e *are invariant with respect to the transformations* (1.11).

PROOF. The first four equations (19.1)b follow at once from (19.1)a. The tensor $H_{\omega\mu\lambda\nu}$ is skew symmetric in $\lambda\nu$, and therefore,

$$H_{\omega\mu 2}{}^{1} = H_{\omega\mu\lambda}{}^{\nu} \underset{2}{a}{}^{\lambda} \overset{1}{a}_{\nu} = H_{\omega\mu\lambda\nu} \underset{2}{a}{}^{\lambda} \underset{2}{a}{}^{\nu} = 0.$$

The remaining equations (19.1)b may be proved in a similar way. Moreover, by virtue of I (8.16)

$$H_{\omega\mu 4}{}^{1} = H_{\omega\mu\lambda}{}^{\nu} \underset{4}{a}{}^{\lambda} \overset{1}{a}_{\nu} = H_{\omega\mu}{}^{\lambda}{}_{\nu} \overset{3}{a}_{\lambda} \underset{2}{a}{}^{\nu} = -H_{\omega\mu\nu}{}^{\lambda} \underset{2}{a}{}^{\nu} \overset{3}{a}_{\lambda} = -H_{\omega\mu 2}{}^{3}.$$

The remaining equations (19.1)c may be proved in the same manner. The vectors $\overset{1}{a}_{\lambda}$ and $\overset{2}{a}_{\lambda}$ being complex conjugates, we have

$$H_{\omega\mu f}{}^{2} = H_{\omega\mu\lambda}{}^{\nu} \underset{f}{a}{}^{\lambda} \overset{2}{a}_{\nu} = H_{\omega\mu\lambda}{}^{\nu} \underset{f}{a}{}^{\lambda} \overset{1}{\bar{a}}_{\nu} = \bar{H}_{\omega\mu f}{}^{1}$$
$$(f = 3, 4).$$

In a similar way we obtain the remaining equations (19.1)d. The equations (19.1)e follow from the last two equations (19.1)c and the first and fourth equations (19.1)b. The invariance of (19.1)b,c,d,e with respect to (1.11) follows easily from (1.11)c,d.

THEOREM (19.1). *The components*

(19.3)a $$\mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}} \overset{\text{def}}{=} 2(\partial_{[\mu} A_{|\mathfrak{a}|}{}^{\mathfrak{b}}{}_{\omega]} + A_{\mathfrak{j}}{}^{\mathfrak{b}}{}_{[\mu} A_{|\mathfrak{a}|}{}^{\mathfrak{j}}{}_{\omega]})$$

define a tensor with respect to $\omega\mu$ *and a* $\mathscr{C}$*-tensor with respect to* a, b. *This tensor is involved in the following identities:*

$$2\mathscr{L}_{[\omega}\mathscr{L}_{\mu]}\, z^{\mathfrak{b}} = -\,\mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}}\, z^{\mathfrak{a}} \tag*{(19.3)b}$$

$$2\mathscr{L}_{[\omega}\mathscr{L}_{\mu]}\, z_{\mathfrak{a}} = \mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}}\, z_{\mathfrak{b}}. \tag*{(19.3)c}$$

The proof follows the pattern of the proof of Theorem IV (1.1).

DEFINITION (19.1). *The tensor* $\mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}}$ *will be referred to as the curvature tensor of* $\mathscr{S}_3$.

THEOREM (19.2). *The curvature tensor* $\mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}}$ *of* $\mathscr{S}_3$ *satisfies the following conditions:*

$$\mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}} = \mathscr{G}_{[\omega\mu]\mathfrak{a}}{}^{\mathfrak{b}} \tag*{(19.4)a}$$

$$\mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{a}} = 0 \tag*{(19.4)b}$$

$$\mathscr{L}_{[\xi}\mathscr{G}_{\omega\mu]\mathfrak{a}}{}^{\mathfrak{b}} = 0. \tag*{(19.5)}$$

PROOF. The equation (19.4)a follows immediately from (19.3)a. From (18.5) we see that

$$\Lambda_{\mathfrak{j}}{}^{\mathfrak{i}}{}_{[\omega}\Lambda_{|\mathfrak{i}|}{}^{\mathfrak{j}}{}_{\mu]} = 0$$

$$\Lambda_{\mathfrak{i}}{}^{\mathfrak{i}}{}_{\mu} = 0.$$

These two equations together with (19.3)a yield (19.4)b. From (19.3)b,c we obtain

$$\begin{aligned} 2\mathscr{L}_{[\xi}\mathscr{L}_{\omega}\mathscr{L}_{\mu]}\, z^{\mathfrak{b}} &= 2\mathscr{L}_{[\xi}(\mathscr{L}_{\omega}\mathscr{L}_{\mu]}\, z^{\mathfrak{b}}) \\ &= -\,(\mathscr{L}_{[\xi}\, z^{\mathfrak{a}})\mathscr{G}_{\omega\mu]\mathfrak{a}}{}^{\mathfrak{b}} - (\mathscr{L}_{[\xi}\mathscr{G}_{\omega\mu]\mathfrak{a}}{}^{\mathfrak{b}})\, z^{\mathfrak{a}}. \end{aligned} \tag*{(19.6)a}$$

On the other hand, we have by virtue of IV (3.4)b

$$\begin{aligned} 2\mathscr{L}_{[\xi}\mathscr{L}_{\omega}\mathscr{L}_{\mu]}\, z^{\mathfrak{b}} &= 2(\mathscr{L}_{[\xi}\mathscr{L}_{\omega})\mathscr{L}_{\mu]}\, z^{\mathfrak{b}} \\ &= H_{[\xi\omega\mu]}{}^{\nu}\mathscr{L}_{\nu}\, z^{\mathfrak{b}} - (\mathscr{L}_{[\mu}\, z^{\mathfrak{a}})\mathscr{G}_{\xi\omega]\mathfrak{a}}{}^{\mathfrak{b}} = -\,(\mathscr{L}_{[\mu}\, z^{\mathfrak{a}})\mathscr{G}_{\xi\omega]\mathfrak{a}}{}^{\mathfrak{b}}. \end{aligned} \tag*{(19.6)b}$$

The equations (19.6)a,b lead at once to (19.5).

THEOREM (19.3). *The tensors* $H_{\omega\mu A}{}^{B}$ *and* $\mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}}$ *are related by the following mutually equivalent relations:*

$$H_{\omega\mu A}{}^{B}\,\delta_{B}^{\mathfrak{a}\mathfrak{b}} = 2\mathscr{G}_{\omega\mu\mathfrak{j}}{}^{[\mathfrak{b}}\,\delta_{A}^{\mathfrak{a}]\mathfrak{j}} \tag*{(19.7)a}$$

$$H_{\omega\mu A}{}^{B}\,\delta_{\mathfrak{a}\mathfrak{b}}^{A} = 2\mathscr{G}_{\omega\mu[\mathfrak{b}}{}^{\mathfrak{j}}\,\delta_{\mathfrak{a}]\mathfrak{j}}^{B} \tag*{(19.7)b}$$

$$H_{\omega\mu\lambda}{}^{\nu}E_{\nu\mathfrak{a}}{}^{\mathfrak{b}} = \mathscr{G}_{\omega\mu\mathfrak{j}}{}^{\mathfrak{b}}E_{\lambda\mathfrak{a}}{}^{\mathfrak{j}} - \mathscr{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{j}}E_{\lambda\mathfrak{j}}{}^{\mathfrak{b}}. \tag*{(19.7)c}$$

PROOF. The equations (18.4) yield

$$\mathscr{L}_{[\omega}\mathscr{L}_{\mu]}\delta^{\mathrm{ab}}_{A} = 0 \tag{19.8)a}$$

$$\mathscr{L}_{[\omega}\mathscr{L}_{\mu]}\delta^{A}_{\mathrm{ab}} = 0 \tag{19.8)b}$$

$$\mathscr{L}_{[\omega}\mathscr{L}_{\mu]}E_{\lambda\mathrm{a}}{}^{\mathrm{b}} = 0. \tag{19.8)c}$$

Using (19.2), (19.3)b,c and IV (3.3)b, one obtains (19.7)a,b,c from (19.8)a,b,c. The equations (18.4)a,b,c are mutually equivalent. Therefore, (19.8)a,b,c are also mutually equivalent.

THEOREM (19.4)a. *The equations* (19.7) *are equivalent to*

$$\mathscr{G}_{\omega\mu 1}{}^{1} = -\mathscr{G}_{\omega\mu 2}{}^{2} = \overline{\mathscr{G}}_{\omega\mu 3}{}^{3} = -\overline{\mathscr{G}}_{\omega\mu 4}{}^{4} = \tfrac{1}{2}(H_{\omega\mu 1}{}^{1} + H_{\omega\mu 3}{}^{3}) \tag{19.9)a}$$

$$\mathscr{G}_{\omega\mu 2}{}^{1} = \overline{\mathscr{G}}_{\omega\mu 4}{}^{3} = H_{\omega\mu 4}{}^{1} \tag{19.9)b}$$

$$\mathscr{G}_{\omega\mu 1}{}^{2} = \overline{\mathscr{G}}_{\omega\mu 3}{}^{4} = H_{\omega\mu 1}{}^{4} \tag{19.9)c}$$

$$\textit{the remaining } \mathscr{G}_{\omega\mu\mathrm{a}}{}^{\mathrm{b}} = 0. \tag{19.9)d}$$

PROOF. The equations (18.4)a are equivalent to

$$\Omega_{B}{}^{A}{}_{\lambda}\,\delta^{\mathrm{ab}}_{A} = 2\Lambda_{\mathrm{j}}{}^{[\mathrm{b}}{}_{\lambda}\,\delta^{\mathrm{a}]\mathrm{j}}_{B}. \tag{19.10}$$

Replacing $\Omega_{A}{}^{B}{}_{\lambda}$ with $H_{\omega\mu A}{}^{B}$ and $\Lambda_{\mathrm{j}\lambda}{}^{\mathrm{b}}$ with $\mathscr{G}_{\omega\mu\mathrm{j}}{}^{\mathrm{b}}$ in (19.10) one obtains (19.7)a. Again if we replace $\Omega_{A}{}^{B}{}_{\lambda}$ with $H_{\omega\mu A}{}^{B}$ in (17.1)b,c,d,e, we obtain (19.1) b,c,d,e. Therefore, if we replace $\Omega_{A}{}^{B}{}_{\lambda}$ with $H_{\omega\mu A}{}^{B}$ and $\Pi_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda}$ with $\mathscr{G}_{\omega\mu\mathrm{a}}{}^{\mathrm{b}}$ in (17.5)b, we obtain the solution $H_{\omega\mu A}{}^{B}$ of (19.7)a. Thus, we have by virtue of (18.5)

$$((\mathscr{G}_{\omega\mu\mathrm{a}}{}^{\mathrm{b}})) = \left(\begin{pmatrix} a_{\omega\mu} & b_{\omega\mu} & 0 & 0 \\ c_{\omega\mu} & -a_{\omega\mu} & 0 & 0 \\ 0 & 0 & \bar{a}_{\omega\mu} & \bar{b}_{\omega\mu} \\ 0 & 0 & \bar{c}_{\omega\mu} & -\bar{a}_{\omega\mu} \end{pmatrix}\right) \tag{19.11}$$

where we must put, in accordance with (17.4)b,c,d,

$$\begin{aligned} 2a_{\omega\mu} &\overset{\text{def}}{=} H_{\omega\mu 1}{}^{1} + H_{\omega\mu 3}{}^{3} \\ b_{\omega\mu} &\overset{\text{def}}{=} H_{\omega\mu 4}{}^{1} \\ c_{\omega\mu} &\overset{\text{def}}{=} H_{\omega\mu 1}{}^{4}. \end{aligned} \tag{19.12}$$

In this way we finally obtain (19.9).

The question arises whether some components $\mathscr{G}_{\omega\mu\mathrm{a}}{}^{\mathrm{b}}$ or a combination of them may be expressed without the help of basic vectors.

The answer to this question is given in the next Theorem where we use the tensors $\overset{xy}{Q}_{\omega\mu}$ and $\underset{1}{E}_\lambda{}^\nu$, $\underset{4}{E}_\lambda{}^\nu$ defined by I (10.6)b and I (10.10).

THEOREM (19.4)b. *We have*

$$(19.13)\text{a} \qquad \mathcal{G}_{\omega\mu 1}{}^{1} = -\tfrac{1}{2}H_{\omega\mu\lambda\nu}(\overset{12}{Q}{}^{\lambda\nu} + \overset{34}{Q}{}^{\lambda\nu})\,[1]$$

$$(19.13)\text{b} \qquad \mathcal{G}_{\omega\mu 2}{}^{1}\mathcal{G}_{\lambda\nu 1}{}^{2} = H_{\omega\mu\alpha}{}^{\beta}H_{\lambda\nu\gamma}{}^{\delta}\underset{1}{E}_{\delta}{}^{\alpha}\underset{4}{E}_{\beta}{}^{\gamma}.$$

PROOF. We derive from the last two equations (19.1)c

$$\begin{aligned} H_{\omega\mu 1}{}^{1} + H_{\omega\mu 3}{}^{3} &= \tfrac{1}{2}(H_{\omega\mu 1}{}^{1} - H_{\omega\mu 2}{}^{2} + H_{\omega\mu 3}{}^{3} - H_{\omega\mu 4}{}^{4}) \\ &= \tfrac{1}{2}(H_{\omega\mu 12} - H_{\omega\mu 21}) + \tfrac{1}{2}(H_{\omega\mu 34} - H_{\omega\mu 43}) \\ &= H_{\omega\mu[12]} + H_{\omega\mu[34]}, \end{aligned}$$

and therefore, by virtue of I (10.8)d

$$(19.14)\text{a} \qquad H_{\omega\mu 1}{}^{1} + H_{\omega\mu 3}{}^{3} = -\,2H_{\omega\mu\lambda\nu}(\overset{12}{Q}{}^{\lambda\nu} + \overset{34}{Q}{}^{\lambda\nu}).$$

On the other hand, we see from I (10.11) that

$$\begin{aligned} (19.14)\text{b} \qquad H_{\omega\mu 1}{}^{4}H_{\lambda\nu 4}{}^{1} &= H_{\omega\mu\alpha}{}^{\beta}H_{\lambda\nu\gamma}{}^{\delta}\underset{1}{a}{}^{\alpha}\overset{4}{a}_{\beta}\underset{4}{a}{}^{\gamma}\overset{1}{a}_{\delta} \\ &= H_{\omega\mu\alpha}{}^{\beta}H_{\lambda\nu\gamma}{}^{\delta}\underset{1}{E}_{\delta}{}^{\alpha}\underset{4}{E}_{\beta}{}^{\gamma}. \end{aligned}$$

The equations (19.13) follow from (19.14) and (19.9).

The following two Theorems will be proved simultaneously.

THEOREM (19.5)a. *In order that there exist a spinor coordinate system for which*

$$(19.15) \qquad \Lambda_{\mathfrak{a}}{}^{\mathfrak{b}}{}_{\lambda} = 0,$$

it is necessary, but not sufficient, that

$$(19.16) \qquad \mathcal{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}} = 0.$$

THEOREM (19.5)b. *A spinor coordinate system for which* (19.15)

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[1] The left-hand term in (19.13)a can be replaced by any one of the components $\mathcal{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}}$ involved in (19.9)a. The left-hand term in (19.13)b can be replaced by a product of any one of the components $\mathcal{G}_{\omega\mu\mathfrak{a}}{}^{\mathfrak{b}}$ involved in (19.9)b and any one of the components involved in (19.9)c.

holds exists if and only if either one of the following mutually equivalent conditions a), b) is satisfied:

a) *The vectors* b_λ, c_λ *involved in* (18.5) *vanish and* a_λ *is a gradient*:

(19.17)a $$\partial_{[\mu}a_{\lambda]} = 0$$

(19.17)b $$b_\lambda = 0$$

(19.17)c $$c_\lambda = 0.$$

b) *The basic vectors fulfill the conditions*

(19.18)a $$(\nabla_{[\lambda}\underset{1}{a}^{\alpha})(\nabla_{\mu]}\overset{1}{a}_{\alpha}) + (\nabla_{[\lambda}\underset{3}{a}^{\alpha})(\nabla_{\mu]}\overset{3}{a}_{\alpha}) = \tfrac{1}{2}(H_{\mu\lambda 1}{}^{1} + H_{\mu\lambda 3}{}^{3})$$

(19.18)b $$\overset{1}{a}_{\alpha}\nabla_\lambda \underset{4}{a}^{\alpha} = 0$$

(19.18)c $$\overset{4}{a}_{\alpha}\nabla_\lambda \underset{1}{a}^{\alpha} = 0.$$

PROOF. According to (18.5) the transformations (18.2) are described by (17.6). Requirement a) of Theorem (19.5)b follows from (18.5) and (17.6). The necessity of (19.16) is obvious. In order to prove its insufficiency we express (19.16) in terms of the vectors a_λ, b_λ, c_λ. Using (18.5) and (19.3)a we obtain

(19.19)a $$\tfrac{1}{2}\mathcal{G}_{\omega\mu 1}{}^{1} = \partial_{[\mu}a_{\omega]} + b_{[\mu}c_{\omega]} = 0$$

(19.19)b $$\tfrac{1}{2}\mathcal{G}_{\omega\mu 1}{}^{2} = \partial_{[\mu}c_{\omega]} + 2c_{[\mu}a_{\omega]} = 0$$

(19.19)c $$\tfrac{1}{2}\mathcal{G}_{\omega\mu 2}{}^{1} = \partial_{[\mu}b_{\omega]} + 2b_{[\omega}a_{\mu]} = 0.$$

The remaining equations (19.16) either reduce to an identity or follow from (19.19). This system can be satisfied by

(19.20) $$c_\omega = \partial_\omega c,\ b_\omega = \partial_\omega c,\ a_\omega = c c_\omega.$$

However, for a suitable choice of c the conditions (19.20) are at variance with (19.17). Hence, (19.16) is not a sufficient condition.

Using (17.4)b,c,d we see that (19.17)b,c are equivalent to (19.18)b,c and that

$$2\partial_{[\mu}a_{\lambda]} = (\nabla_{[\lambda}\underset{1}{a}^{\alpha})(\nabla_{\mu]}\overset{1}{a}_{\alpha}) + (\nabla_{[\lambda}\underset{3}{a}^{\alpha})(\nabla_{\mu]}\overset{3}{a}_{\alpha})$$
$$+ \overset{1}{a}_{\alpha}\nabla_{[\mu}\nabla_{\lambda]}\underset{1}{a}^{\alpha} + \overset{3}{a}_{\alpha}\nabla_{[\mu}\nabla_{\lambda]}\underset{3}{a}^{\alpha}.$$

Therefore this equation is equivalent by virtue of IV (3.3)a to

$$(19.21) \qquad 2\partial_{[\mu} a_{\lambda]} = (\nabla_{[\lambda} \underset{1}{a^{\alpha}}) (\nabla_{\mu]} \overset{1}{a}_{\alpha}) + (\nabla_{[\lambda} \underset{3}{a^{\alpha}}) (\nabla_{\mu]} \overset{3}{a}_{\alpha})$$

$$- \tfrac{1}{2}(H_{\mu\lambda 1}{}^{1} + H_{\mu\lambda 3}{}^{3}).$$

The remaining part of the proof is obvious.

NOTE. If (19.18)b,c hold then according to (19.9)a and (19.19)a

$$(19.22) \qquad \mathscr{G}_{\lambda\mu 1}{}^{1} = 2\partial_{[\mu} a_{\lambda]} = \tfrac{1}{2}(H_{\lambda\mu 1}{}^{1} + H_{\lambda\mu 3}{}^{3})$$

so that (19.21) reduces to

$$(\nabla_{[\lambda} \underset{1}{a^{\alpha}}) (\nabla_{\mu]} \overset{1}{a}_{\alpha}) + (\nabla_{[\lambda} \underset{3}{a^{\alpha}}) (\nabla_{\mu]} \overset{3}{a}_{\alpha}) = 0$$

and (19.18)a yields $\mathscr{G}_{\lambda\mu 1}{}^{1} = 0$ by virtue of (19.22). The remaining equations (19.16) are satisfied by virtue of (19.18)b,c.

The equation (19.16) has an interesting physical interpretation.

THEOREM (19.6). *A necessary and sufficient condition that the electromagnetic field of the unified tensor field* $g_{\lambda\mu}$ *generate no gravitation*[1] *is that* (19.16) *be satisfied.*

PROOF. Assume that the electromagnetic field of $g_{\lambda\mu}$ does not generate gravitation. Then we must have

$$(19.23) \qquad H_{\omega\mu\lambda}{}^{\nu} = 0.$$

The equation (19.16) follows from (19.23) and (19.9). In order to prove the sufficiency, first introduce the symbols $\chi_a{}^b$ for $H_{\omega\mu a}{}^{b}$. Then we have according to (19.1)b,c,d,e

$$(19.24) \qquad \left(\left(\begin{matrix} \chi_1{}^1 \cdots \chi_4{}^1 \\ \vdots \qquad \vdots \\ \chi_1{}^4 \cdots \chi_4{}^4 \end{matrix}\right)\right) = \left(\left(\begin{matrix} \chi_1{}^1 & 0 & \chi_3{}^1 & -\bar{\chi}_1{}^3 \\ 0 & -\chi_1{}^1 & \bar{\chi}_3{}^1 & -\chi_1{}^3 \\ \chi_1{}^3 & \bar{\chi}_1{}^3 & \chi_3{}^3 & 0 \\ -\bar{\chi}_3{}^1 & -\chi_3{}^1 & 0 & -\chi_3{}^3 \end{matrix}\right)\right).$$

Assume now that (19.16) holds. Then $\mathscr{G}_{\omega\mu 1}{}^{1}$ and $\bar{\mathscr{G}}_{\omega\mu 1}{}^{1}$ vanish so that we have according to (19.9)a and (19.1)c,d

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[1] Such tensors exist. *Cf.*, for instance, Theorem V (9.2).

$$\chi_1{}^1 + \chi_3{}^3 = 0$$

$$\bar{\chi}_1{}^1 + \chi_3{}^3 = \chi_2{}^2 + \chi_3{}^3 = -\chi_1{}^1 + \chi_3{}^3 = 0.$$

Hence,

(19.25)a $$\chi_1{}^1 = \chi_3{}^3 = 0.$$

Moreover, we have by virtue of (19.16), (19.9)c and (19.1)c,d

(19.25)b₁ $$0 = \chi_1{}^4 = \bar{\chi}_2{}^4 = -\bar{\chi}_3{}^1$$

and, therefore, also

(19.25)b₂ $$\chi_3{}^1 = 0.$$

Using again (19.1)c,d and (19.16) we obtain by virtue of (19.9)b

(19.25)c₁ $$0 = \chi_4{}^1 = \bar{\chi}_4{}^2 = -\bar{\chi}_1{}^3,$$

and therefore,

(19.25)c₂ $$\chi_1{}^3 = 0.$$

Hence, all terms of the matrix (19.24) are equal to zero, so that the equation (19.23) holds. Therefore, the electromagnetic field of $g_{\lambda\mu}$ does not generate gravitation.

NOTE. Even in the absence of gravitation our space-time is not a Minkowskian one, because its curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ does not vanish [1]. Thus, if $U \neq O$ are two distinct points of our space-time, we have $T_4(U) \neq T_4(O)$, and therefore, $\mathscr{S}_3(U) \neq \mathscr{S}_3(O)$. *Thus, $\mathscr{S}_3$ does not, in general, reduce to $\mathscr{S}_3(O)$ as in the classical special theory of relativity.* This accounts for the fact that we may have $\Lambda_{\mathrm{a}}{}^{\mathrm{b}}{}_{\lambda} \neq 0$ even if (19.16) holds.

20. Dirac's and Schrödinger's equations in $\mathscr{S}_3$. In section 14 we derived the equations (14.8) and (14.17) for the case of the special theory of relativity. In fact, we based our considerations on the metric (14.1) and introduced the electromagnetism separately through (14.5)a. In this section we shall find equations in $\mathscr{S}_3$ which reduce for the case of special relativity to (14.8) and (14.17). To this effect we shall first prove the following

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[1] *Cf.* IV (3.10) where in the absence of gravitation we have $H_{\omega\mu\lambda}{}^{\nu} = 0$.

LEMMA (20.1). *Put*

$$(20.1)\text{a}\qquad \begin{aligned} \sqrt{2}\underset{1}{u}^\nu &= \underset{1}{a}^\nu + \underset{2}{a}^\nu, & \sqrt{2}i\underset{2}{u}^\nu &= \underset{2}{a}^\nu - \underset{1}{a}^\nu, \\ \sqrt{2}\underset{3}{u}^\nu &= \underset{3}{a}^\nu + \underset{4}{a}^\nu, & \sqrt{2}\underset{4}{u}^\nu &= \underset{3}{a}^\nu - \underset{4}{a}^\nu. \end{aligned}$$

Then $\underset{1}{u}^\nu, \ldots, \underset{4}{u}^\nu$ *are mutually perpendicular unit vectors and, in particular,*

$$(20.1)\text{b}\qquad \underset{1}{u}_\lambda \underset{1}{u}^\lambda = \underset{2}{u}_\lambda \underset{2}{u}^\lambda = \underset{3}{u}_\lambda \underset{3}{u}^\lambda = -\underset{4}{u}_\lambda \underset{4}{u}^\lambda = 1.$$

Put

$$(20.2)\qquad \underset{j}{u}^\lambda \overset{i}{u}_\lambda = \delta^i_j.$$

Then

$$(20.3)\qquad \overset{1}{u}_\lambda = \underset{1}{u}_\lambda,\quad \overset{2}{u}_\lambda = \underset{2}{u}_\lambda,\quad \overset{3}{u}_\lambda = \underset{3}{u}_\lambda,\quad \overset{4}{u}_\lambda = -\underset{4}{u}_\lambda$$

and

$$(20.4)\qquad \begin{aligned} \sqrt{2}\overset{1}{a}_\lambda &= \overset{1}{u}_\lambda + i\overset{2}{u}_\lambda \\ \sqrt{2}\overset{2}{a}_\lambda &= \overset{1}{u}_\lambda - i\overset{2}{u}_\lambda \\ \sqrt{2}\overset{3}{a}_\lambda &= \overset{3}{u}_\lambda + \overset{4}{u}_\lambda \\ \sqrt{2}\overset{4}{a}_\lambda &= \overset{3}{u}_\lambda - \overset{4}{u}_\lambda. \end{aligned}$$

PROOF. The properties of the vectors $\underset{j}{u}^\nu$ follow easily from (20.1)a and from Theorem I (6.3). The vectors $\underset{j}{u}^\nu$ being linearly independent, there is only one system $\overset{i}{u}_\lambda$ satisfying (20.2). This fact leads to (20.3) by virtue of (20.1)b. Solving (20.1)a for the $\underset{j}{a}^\nu$ we obtain

$$\begin{aligned} \sqrt{2}\underset{2}{a}^\nu &= \underset{1}{u}^\nu + i\underset{2}{u}^\nu, & \sqrt{2}\underset{1}{a}^\nu &= \underset{1}{u}^\nu - i\underset{2}{u}^\nu \\ \sqrt{2}\underset{4}{a}^\nu &= \underset{3}{u}^\nu - \underset{4}{u}^\nu, & \sqrt{2}\underset{3}{a}^\nu &= \underset{3}{u}^\nu + \underset{4}{u}^\nu . \end{aligned}$$

These equations together with (20.3) and I (8.16) lead to (20.4).

THEOREM (20.1). *Put*

$$v^a \overset{\text{def}}{=} h^{\lambda\mu} \overset{a}{a}_\lambda \left(\frac{\hbar}{i} \mathscr{L}_\mu + A_\mu\right) \tag{20.5}$$

and define ${}_*v_{\mathbf{a}}{}^{\mathbf{b}}$ *by means of* (13.6)a. *The equations*

$$ {}_*v_{\mathbf{a}}{}^{\mathbf{b}} \mathscr{X}^{\mathbf{a}} + i\mu c \mathscr{X}^{\mathbf{b}} = 0 \tag{20.6}$$

$$({}_*\omega + \mu^2 c^2)\zeta = 0 \tag{20.7}$$

$$({}_*\omega \overset{\text{def}}{=} h_{ab} v^a v^b)$$

have the following properties:

1) *They are independent of the transformations* (5.1), (5.7), (1.11) *and of the transformations of the coordinates* x^ν.

2) *They reduce to Dirac's equations* (14.8) *and Schrödinger's equations* (14.17) [1] *under the following restrictions:*

a) *The tensor* $h_{\lambda\mu}$ *is given by*

$$((h_{\lambda\mu})) = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^2 \end{pmatrix}\right), \tag{20.8}$$

and the coordinates $x^{\mathrm{I}}, \ldots, x^{\mathrm{IV}}$ *are identified with* x, y, z, t.

b) *The vector* A_λ *expressed in these coordinates is identified with the vector* (A_x, A_y, A_z, A_t) *involved in* (14.5)a.

c) *The equations* (19.15) *hold.*

PROOF. The statement 1) follows at once from (20.5), (20.6) and (20.7). In order to prove the statement 2) we shall rewrite the equations (14.8) in a form suitable for generalization. First we identify x, y, z, t with $x^{\mathrm{I}}, \ldots, x^{\mathrm{IV}}$ so that we have (20.8) instead of (14.1). We shall also write A_λ for the components $A_x, \ldots, A_t$ involved in (14.5)a. Then we introduce the vectors

$$\begin{aligned} &\underset{1}{u}^\alpha(1, 0, 0, 0), && \overset{1}{u}_\alpha(1, 0, 0, 0) \\ &\underset{2}{u}^\alpha(0, 1, 0, 0), && \overset{2}{u}_\alpha(0, 1, 0, 0) \\ &\underset{3}{u}^\alpha(0, 0, 1, 0), && \overset{3}{u}_\alpha(0, 0, 1, 0) \\ &\underset{4}{u}^\alpha(0, 0, 0, c^{-1}), && \overset{4}{u}_\alpha(0, 0, 0, c). \end{aligned} \tag{20.9}$$

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[1] with the generalized operator (14.7)b. This generalization is enforced by the tensor $g_{\lambda\mu}$ of the unified field theory.

The vectors $\overset{i}{u}{}^{\nu}$ ($\underset{j}{u}{}_{\nu}$) are mutually perpendicular unit vectors. Both sets (20.9) satisfy (20.2), (20.3) and (20.1)b. Next we write $d_1, \ldots, d_4$ and $d^1, \ldots, d^4$ instead of $d_x, \ldots, d_t$ and $d^x, \ldots, d^t$. Thus, we have according to (14.5)a and (14.7)a

$$\text{(20.10)} \qquad d_v = \underset{v}{u}{}^{\lambda}\left(\frac{\hbar}{i}\,\partial_\lambda + A_\lambda\right), \quad c^{-1}d_4 = \underset{4}{u}{}^{\lambda}\left(\frac{\hbar}{i}\,\partial_\lambda + A_\lambda\right)$$

$$d^v = h^{\lambda\mu}\overset{v}{u}{}_{\mu}\left(\frac{\hbar}{i}\,\partial_\lambda + A_\lambda\right), \quad cd^4 = h^{\lambda\mu}\overset{4}{u}{}_{\mu}\left(\frac{\hbar}{i}\,\partial_\lambda + A_\lambda\right)$$

$$(v = 1, 2, 3),$$

and therefore, by virtue of (14.7)b

$$\text{(20.11)} \qquad \sqrt{2}v^1 = h^{\lambda\mu}(\overset{1}{u}{}_{\mu} + i\overset{2}{u}{}_{\mu})\left(\frac{\hbar}{i}\,\partial_\lambda + A_\lambda\right)$$

$$\sqrt{2}v^2 = h^{\lambda\mu}(\overset{1}{u}{}_{\mu} - i\overset{2}{u}{}_{\mu})\left(\frac{\hbar}{i}\,\partial_\lambda + A_\lambda\right)$$

$$\sqrt{2}v^3 = h^{\lambda\mu}(\overset{3}{u}{}_{\mu} + \overset{4}{u}{}_{\mu})\left(\frac{\hbar}{i}\,\partial_\lambda + A_\lambda\right)$$

$$\sqrt{2}v^4 = h^{\lambda\mu}(\overset{3}{u}{}_{\mu} - \overset{4}{u}{}_{\mu})\left(\frac{\hbar}{i}\,\partial_\lambda + A_\lambda\right).$$

The vector v^a is transformed according to (1.11)e if we perform a rotation of $\overset{1}{u}{}_{\mu}$ and $\overset{2}{u}{}_{\mu}$ (in the plane $\overset{1}{u}{}_{[\mu}\overset{2}{u}{}_{\lambda]}$) through the angle $-\varphi$ and a Minkowskian rotation of $\overset{3}{u}{}_{\mu}$ and $\overset{4}{u}{}_{\mu}$ (in the plane $\overset{3}{u}{}_{[\mu}\overset{4}{u}{}_{\lambda]}$) through the angle $-\psi$.[1] Therefore, we may take for $\overset{1}{u}{}_{\mu}, \ldots, \overset{4}{u}{}_{\mu}$ in (20.11) any quadruple of vectors which one obtains from the second set (20.9) by the above rotations. Then the equations (20.11) are starting points of the generalization we are looking for if dealing with the unified tensor field $g_{\lambda\mu}$. In this case we know the basic vectors $\overset{i}{a}{}^{\nu}$, $\underset{j}{a}{}_{\nu}$ of the space-time. Thus, we may apply (20.4) to

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[1] Followed by a reflection if $\epsilon' = -1$ in (1.11)c,d.

(20.11) and obtain

$$v^a = h^{\lambda\mu}\overset{a}{a}_{\mu}\left(\frac{\hbar}{i}\,\partial_\lambda + A_\lambda\right). \tag{20.12}$$

The tensor $h_{\lambda\mu}$ which appears in (20.12) is now to be defined by $h_{\lambda\mu} \overset{\text{def}}{=} g_{(\lambda\mu)}$. In order to make the right-hand term in (20.12) a scalar operator with respect to arbitrary transformations of coordinates in our space-time, we have to replace ∂_λ by $\mathscr{L}_\lambda$. In doing so we obtain (20.5) from (20.12). Then the proof of our Theorem follows easily.

NOTE. Another set of equations similar to Dirac's equations may be derived from the conservation law V (16.9)b [1] which in non-holonomic components has the form

$$\nabla_i \mathfrak{A} w^i = 0. \,^{2} \tag{20.13}$$

Let $\mathscr{X}^{\mathrm{a}}, y^{\mathrm{a}}$ be two arbitrary spinors incident with one of the spinor lines ${}^{\epsilon}w^{\mathrm{ab}}$ representing in $\mathscr{S}_3$ the vector w^i. Then we have according to (2.8)b

$$\sqrt{2}\mathfrak{A} w^i = \delta^{i}_{\mathrm{ab}}\mathscr{X}^{\mathrm{a}} y^{\mathrm{b}}\mathfrak{A}. \tag{20.14}$$

Applying to this equation the operator

$$\mathscr{L}_i \overset{\text{def}}{=} \underset{i}{a}^{\lambda}\mathscr{L}_\lambda$$

we obtain according to (18.9)c

$$\sqrt{2}\mathscr{L}_i \mathfrak{A} w^i = \delta^{i}_{\mathrm{ab}}\mathscr{L}_i \mathscr{X}^{\mathrm{a}} y^{\mathrm{b}}\mathfrak{A}. \tag{20.15)a}$$

The first term of (20.15)a vanishes because of (20.13):

$$\mathscr{L}_i \mathfrak{A} w^i = \nabla_i \mathfrak{A} w^i = 0$$

so that (20.15)a reduces to

$$\delta^{i}_{\mathrm{ab}} y^{\mathrm{b}}\mathscr{L}_i \mathscr{X}^{\mathrm{a}} + \delta^{i}_{\mathrm{ab}}\mathscr{X}^{\mathrm{a}}\mathscr{L}_i y^{\mathrm{b}} = 0. \tag{20.15)b}$$

Put

$$ {}^{+}\mathscr{X}^{\mathrm{a}} \overset{\text{def}}{=} \mathscr{X}^{\mathrm{a}}, \quad {}^{-}\mathscr{X}^{\mathrm{a}} \overset{\text{def}}{=} y^{\mathrm{a}}.$$

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[1] See Infeld, L., and v. d. Waerden, B., (1), where Dirac's equations are found in two-component spinor analysis associate with the gravitational theory.

[2] $\mathfrak{A}$ is defined by I (8.8)a. Hence, $\nabla_i \mathfrak{A} = 0$.

We have according to (7.9)

$${}^{+}\mathscr{X}_{\mathrm{a}}\,{}^{-}\mathscr{X}^{\mathrm{a}} + {}^{+}\mathscr{X}^{\mathrm{a}}\,{}^{-}\mathscr{X}_{\mathrm{a}} = 0.$$

Thus, the equation (20.15)b is satisfied if

$$\text{(20.15)c} \qquad \delta^{i}_{\mathrm{ab}}\mathscr{L}_{i}\,{}^{\epsilon}\mathscr{X}^{\mathrm{a}} = -\,\epsilon(\rho\,{}^{\epsilon}\mathscr{X}_{\mathrm{b}} + A_{i}\delta^{i}_{\mathrm{ab}}\,{}^{\epsilon}\mathscr{X}^{\mathrm{a}}), \quad \epsilon = \pm\,1.$$

Here ρ is an arbitrary scalar, A_i an arbitrary vector and ${}^{\epsilon}\mathscr{X}$ is either ${}^{+}\mathscr{X}$ or ${}^{-}\mathscr{X}$.[1] Put

$$\sqrt{2}\,{}^{\epsilon}L^{i} \overset{\mathrm{def}}{=} h^{ij}(\mathscr{L}_{j} + \epsilon A_{j})$$

$$\text{(20.16)} \qquad (({}^{\epsilon}\mathscr{L}_{\mathrm{a}}{}^{\mathrm{b}})) \overset{\mathrm{def}}{=} i\left(\begin{pmatrix} 0 & 0 & {}^{\epsilon}L^{1} & {}^{\epsilon}L^{3} \\ 0 & 0 & {}^{\epsilon}L^{4} & -\,{}^{\epsilon}L^{2} \\ -\,{}^{\epsilon}L^{2} & -\,{}^{\epsilon}L^{3} & 0 & 0 \\ -\,{}^{\epsilon}L^{4} & {}^{\epsilon}L^{1} & 0 & 0 \end{pmatrix}\right).$$ [2]

A straightforward computation shows that (20.15)c is equivalent to

$$\text{(20.17)} \qquad {}^{\epsilon}\mathscr{L}_{\mathrm{a}}{}^{\mathrm{b}}\,{}^{\epsilon}\mathscr{X}^{\mathrm{a}} + \epsilon\rho i\,{}^{\epsilon}\mathscr{X}^{\mathrm{b}} = 0.$$

This equation is of the type (20.6).

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[1] ϵ in ${}^{\epsilon}\mathscr{X}^{\mathrm{a}}$ is not connected with ϵ in ${}^{\epsilon}w^{\mathrm{ab}} = {}^{+}\mathscr{X}^{[\mathrm{a}}\,{}^{-}\mathscr{X}^{\mathrm{b}]}$.

[2] *Cf.* (13.6)b.

APPENDIX III[1]

LINE GEOMETRY IN A THREE-DIMENSIONAL PROJECTIVE SPACE

1. Introduction. In this appendix we shall display some elementary results of line geometry which are needed for a better understanding of the first and sixth chapters. More information about line geometry can be found in Hlavatý, V., (11).

Let $c_\lambda{}^{\nu'}$ be a set of real constants with the property

$$(1.1) \qquad \varDelta \overset{\text{def}}{=} \operatorname{Det}((c_\lambda{}^{\nu'})) = 1.$$

Consider the group of projective transformations

$$(1.2) \qquad x^{\nu'} = c_\lambda{}^{\nu'} x^\lambda, \quad x^\lambda = c_{\nu'}{}^\lambda x^{\nu'}$$

of projective homogeneous coordinates x^ν which describe a three-dimensional projective space P_3. The group (1.2) is the basis for the usual definition of tensors by means of the transformation rule

$$T^{\nu'\cdots}_{\lambda'\cdots} = c_\alpha{}^{\nu'} \ldots c_{\lambda'}{}^\beta \ldots T^{\alpha\cdots}_{\beta\cdots}.$$

Throughout this appendix we shall consider $T^{\nu\cdots}_{\lambda\cdots}$ and $\rho T^{\nu\cdots}_{\lambda\cdots}$ as components of the same tensor T.[2] Here ρ is an arbitrary real factor of proportionality. In other words, we regard $T^{\nu\cdots}_{\lambda\cdots}$ as homogeneous components of a tensor T. In particular, x^ν are homogeneous coordinates of a contravariant vector, which we call a point. Homogeneous coordinates y_λ of a covariant vector define a plane.

Because of (1.1) the space P_3 does not possess any densities.[3] Therefore, the indicators defined by Definition I (2.1) behave like tensors. We shall denote them by $I^{\omega\mu\lambda\nu}$ and $i_{\omega\mu\lambda\nu}$. Besides δ^ν_λ, indicators are the only tensors whose components are not considered homogeneous. In other words, $I^{\omega\mu\lambda\nu}$ and $\rho I^{\omega\mu\lambda\nu}$ are to be looked upon as components of two different tensors. This exception to our rule about homogeneous coordinates does not restrict the generality and is very convenient for our future considerations.

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[1] Greek indices run here from I to IV.

[2] The only exception to this agreement will be mentioned shortly.

[3] We mean by this that a scalar density cannot be distinguished from a scalar. A similar remark holds for tensor densities.

In this appendix we shall deal almost exclusively with scalars, vectors, skew symmetric quadratic tensors and indicators. The indicators are a valuable tool in our considerations as is shown in the following three examples.

1) Let y^ν, z^ν, t^ν be three linearly independent points. Then

$$x_\omega \overset{\text{def}}{=} i_{\omega\mu\lambda\nu} y^\mu z^\lambda t^\nu \tag{1.3)a}$$

is a plane incident with y^ν, z^ν and t^ν. Similarly, if y_λ, z_λ, t_λ are three linearly independent planes, then

$$x^\omega \overset{\text{def}}{=} I^{\omega\mu\lambda\nu} y_\mu z_\lambda t_\nu \tag{1.3)b}$$

is a point incident with y_λ, z_λ, t_λ.

2) Indicators will be used for raising and/or lowering *both* indices of skew symmetric quadratic tensors in the following manner:

$$\begin{gathered} t_{\omega\mu} = \tfrac{1}{2} i_{\omega\mu\lambda\nu} t^{\lambda\nu} = \tfrac{1}{2} i_{\lambda\nu\omega\mu} t^{\lambda\nu} \\ t^{\omega\mu} = \tfrac{1}{2} I^{\omega\mu\lambda\nu} t_{\lambda\nu} = \tfrac{1}{2} I^{\lambda\nu\omega\mu} t_{\lambda\nu} \\ (t_{\omega\mu} = t_{[\omega\mu]}). \end{gathered} \tag{1.4}$$

Then, of course,

$$t_{\omega\mu} t^{\omega\mu} = \tfrac{1}{2} i_{\omega\mu\lambda\nu} t^{\omega\mu} t^{\lambda\nu} = \tfrac{1}{2} I^{\omega\mu\lambda\nu} t_{\omega\mu} t_{\lambda\nu}. \tag{1.5}$$

3) Denote by t the determinant of a tensor $t_{\omega\mu} = t_{[\omega\mu]}$. Then

$$t = (t_{I\,II} t_{III\,IV} + t_{II\,III} t_{I\,IV} + t_{III\,I} t_{II\,IV})^2, \tag{1.6)a}$$

and according to (1.5) and Theorem I (2.3) (for $t_{\omega\mu} = k_{\omega\mu}$),

$$\begin{aligned} 64t &= (I^{\omega\mu\lambda\nu} t_{\omega\mu} t_{\lambda\nu})^2 \\ &= (i_{\omega\mu\lambda\nu} t^{\omega\mu} t^{\lambda\nu})^2 \\ &= 4(t_{\omega\mu} t^{\omega\mu})^2. \end{aligned} \tag{1.6)b}$$

2. Plücker coordinates of a line.

DEFINITION (2.1). *Let y^ν, z^ν (y_λ, z_λ) be two distinct points on a line L (two distinct planes through a line L). The determinants*

$$y^{[\lambda} z^{\nu]}, \quad (y_{[\lambda} z_{\nu]}) \tag{2.1}$$

are called projective homogeneous point (plane) Plücker coordinates of L.

THEOREM (2.1). *Put*

$$p^{\lambda\nu} \overset{\text{def}}{=} y^{[\lambda} z^{\nu]}, \quad q_{\lambda\nu} \overset{\text{def}}{=} y_{[\lambda} z_{\nu]}. \tag{2.2}$$

$p^{\lambda\nu}(q_{\lambda\nu})$ *is a tensor with the following properties:*

a) *It does not depend on the choice of the points* y^ν, $z^\nu \neq y^\nu$ (*planes* y_λ, $z_\lambda \neq y_\lambda$) *incident with L.*

b) $p^{\lambda\nu}$ *and* $q^{\lambda\nu}$ *are related by*

$$q^{\omega\mu} = \tfrac{1}{2} I^{\omega\mu\lambda\nu} q_{\lambda\nu} = \rho p^{\omega\mu} \tag{2.3)a}$$

$$p_{\omega\mu} = \tfrac{1}{2} i_{\omega\mu\lambda\nu} p^{\lambda\nu} = \sigma q_{\omega\mu} \tag{2.3)b}$$

$$(\rho\sigma \neq 0).$$

c) $p^{\lambda\nu}$ *and* $q_{\lambda\nu}$ *satisfy the requirements*

$$p_{\omega\mu} p^{\omega\mu} = 0 \tag{2.4)a}$$

$$q_{\omega\mu} q^{\omega\mu} = 0. \tag{2.4)b}$$

PROOF. Introduce the points

$$'y^\nu = \alpha y^\nu + \beta z^\nu, \quad 'z^\nu = \gamma y^\nu + \delta z^\nu \tag{2.5)a}$$

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0 \tag{2.5)b}$$

and the tensor

$$'p^{\lambda\nu} \overset{\text{def}}{=} {'y}^{[\lambda}\, {'z}^{\nu]}. \tag{2.6}$$

The points $'y^\nu$ and $'z^\nu$ are two distinct points on L. The equations (2.2), (2.5) and (2.6) yield

$$'p^{\lambda\nu} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} p^{\lambda\nu}.$$

This equation proves statement a) concerning the tensor $p^{\lambda\nu}$. The same statement concerning $q_{\lambda\nu}$ can be proved in a similar manner.

Let a^ν, b^ν be two arbitrary points such that a^ν, b^ν, y^ν, z^ν are linearly independent. Then

$$\rho \overset{\text{def}}{=} i_{\omega\mu\lambda\nu} a^\omega b^\mu y^\lambda z^\nu \neq 0. \tag{2.7}$$

Therefore, the planes

$$\begin{aligned} a_\omega &\overset{\text{def}}{=} i_{\omega\mu\lambda\nu} a^\mu p^{\lambda\nu} = i_{\omega\mu\lambda\nu} a^\mu y^\lambda z^\nu \\ b_\omega &\overset{\text{def}}{=} i_{\omega\mu\lambda\nu} b^\mu p^{\lambda\nu} = i_{\omega\mu\lambda\nu} b^\mu y^\lambda z^\nu \end{aligned} \tag{2.8}$$

exist and are incident with L. We may write, according to statement a),

$$q_{\lambda\nu} = a_{[\lambda}b_{\nu]}. \tag{2.9}$$

Put

$$q^{\omega\mu} = \tfrac{1}{2}I^{\omega\mu\lambda\nu}q_{\lambda\nu}. \tag{2.10)a}$$

Substituting into this equation from (2.8) and (2.9), and remembering that the equations I (2.2) also hold for $I^{\omega\mu\lambda\nu}$ and $i_{\omega\mu\lambda\nu}$, we obtain

$$q^{\omega\mu} = (a^{\omega}p^{\mu\nu} + a^{\nu}p^{\omega\mu} + a^{\mu}p^{\nu\omega})i_{\nu\alpha\beta\gamma}b^{\alpha}p^{\beta\gamma}. \tag{2.10)b}$$

On the other hand,

$$\begin{aligned} i_{\nu\alpha\beta\gamma}p^{\beta\gamma}p^{\lambda\nu} &= i_{\nu\alpha\beta\gamma}y^{\beta}z^{\gamma}y^{[\lambda}z^{\nu]} \\ &= \tfrac{1}{2}[y^{\lambda}i_{\nu\alpha\beta\gamma}y^{\beta}z^{\gamma}z^{\nu} - z^{\lambda}i_{\nu\alpha\beta\gamma}y^{\beta}y^{\nu}z^{\gamma}] = 0. \end{aligned}$$

Therefore, (2.10)b reduces, by virtue of (2.7), to (2.3)a. The equation (2.3)b can be proved in a similar manner.

The equation (2.4)a follows from

$$\begin{aligned} p_{\omega\mu}p^{\omega\mu} &= \tfrac{1}{2}i_{\omega\mu\lambda\nu}p^{\omega\mu}p^{\lambda\nu} \\ &= \tfrac{1}{2}i_{\omega\mu\lambda\nu}y^{\omega}z^{\mu}y^{\lambda}z^{\nu} = 0. \end{aligned}$$

The equation (2.4)b can be proved in a similar manner.

THEOREM (2.2). *A necessary and sufficient condition that* $p^{\lambda\nu} = p^{[\lambda\nu]}$ *be Plücker coordinates of a line* L *is that* (2.4)a *be satisfied. If this requirement is fulfilled, the tensor* $p^{\lambda\nu}$ *determines a unique line* L *whose coordinates are* $p^{\lambda\nu}$, $p_{\lambda\nu}$.

PROOF. The necessity of (2.4)a has already been proved. Assume that (2.4)a holds, and denote by p the determinant of $p_{\lambda\nu}$. Then we have, according to (2.4)a and (1.6)b, $p = 0$. Therefore, the rank r of the matrix $((p_{\lambda\nu}))$ is $r < 4$. Because $p_{\lambda\nu}$ is skew symmetric, we cannot have $r = 1, 3$. Hence, $r = 2$. It is known from elementary algebra that in this case $p_{\lambda\nu}$ is of the form $p_{\lambda\nu} = y_{[\lambda}z_{\nu]}$. Thus, $p_{\lambda\nu}$ determines a unique line L as the intersection of y_{λ} and z_{λ}. Its coordinates are $p_{\lambda\nu}$ as well as $p^{\lambda\nu}$.

In the next Theorem we denote by $\underset{\circ}{\nu}$ one arbitrary but fixed symbol of the four symbols I, ..., IV.

THEOREM (2.3). *A necessary and sufficient condition that a line*

$p^{\lambda\nu}$ be incident with the coordinate plane

(2.11) $$x^{\underset{\circ}{\nu}} = 0$$

is

(2.12) $$p^{\lambda\underset{\circ}{\nu}} = 0.$$

If this requirement is not fulfilled, then the line $p^{\lambda\nu}$ intersects the plane in the point $p^{\lambda\underset{\circ}{\nu}}$.

PROOF. Put $\underset{\circ}{\nu} = \mathrm{IV}$ and assume that the line $p^{\lambda\nu} = y^{[\lambda}z^{\nu]}$ is incident with the plane (2.11). Then we have $y^{\mathrm{IV}} = z^{\mathrm{IV}} = 0$, and consequently, (2.12) holds for $\underset{\circ}{\nu} = \mathrm{IV}$. Assume that (2.12) holds for $\underset{\circ}{\nu} = \mathrm{IV}$. Then

$$a_\lambda p^{\lambda\nu} = 0$$

where $a_\lambda(0, 0, 0, 1)$ are the coordinates of the plane (2.11) for $\underset{\circ}{\nu} = \mathrm{IV}$. Hence, $p^{\lambda\nu}$ is incident with this plane. The proof for $\underset{\circ}{\nu} = \mathrm{I}, \mathrm{II}, \mathrm{III}$ follows a similar pattern. Assume now that

(2.13) $$p^{\lambda\underset{\circ}{\nu}} \neq 0;$$

put $\underset{\circ}{\nu} = \mathrm{IV}$ and denote by y^ν the intersection point of $p^{\lambda\nu}$ and the plane (2.11). Then $y^{\mathrm{IV}} = 0$ and

$$p^{\lambda\,\mathrm{IV}} = y^{[\lambda}z^{\mathrm{IV}]} = \tfrac{1}{2}y^\lambda z^{\mathrm{IV}}.$$

According to (2.13) the line $p^{\lambda\nu}$ is not incident with (2.11) for $\underset{\circ}{\nu} = \mathrm{IV}$, and therefore, $z^{\mathrm{IV}} \neq 0$. Thus,

$$p^{\mathrm{I\,IV}} : p^{\mathrm{II\,IV}} : p^{\mathrm{III\,IV}} : p^{\mathrm{IV\,IV}} = y^{\mathrm{I}} : y^{\mathrm{II}} : y^{\mathrm{III}} : 0.$$

The proof for $\underset{\circ}{\nu} = \mathrm{I}, \mathrm{II}, \mathrm{III}$ follows a similar pattern.

3. Two and more lines.

THEOREM (3.1). *A necessary and sufficient condition that two lines $\underset{a}{p}^{\lambda\nu}$ $(a = 1, 2)$ intersect is*

(3.1) $$\underset{a}{p}_{\lambda\nu}\underset{b}{p}^{\lambda\nu} = 0 \ \ (a, b = 1, 2).$$

PROOF. Put

(3.2) $$\underset{1}{p}^{\lambda\nu} = x^{[\lambda}y^{\nu]}, \ \underset{2}{p}^{\lambda\nu} = z^{[\lambda}t^{\nu]}.$$

A necessary and sufficient condition for the lines to intersect is that the determinant $|x^\lambda, y^\lambda, z^\lambda, t^\lambda|$ vanish. This condition is equivalent to

$$0 = i_{\omega\mu\lambda\nu} x^\omega y^\mu z^\lambda t^\nu = i_{\omega\mu\lambda\nu} \underset{1}{p}^{\omega\mu} \underset{2}{p}^{\lambda\nu} = 2 \underset{1}{p}_{\lambda\nu} \underset{2}{p}^{\lambda\nu}.$$

The remaining equations (3.1) follow from the fact that $\underset{a}{p}^{\lambda\nu}$ $(a = 1, 2)$ are lines.

THEOREM (3.2). *The pencil P of lines $r^{\lambda\nu}$*[1] *determined by two intersecting lines $\underset{a}{p}^{\lambda\nu}$ $(a = 1, 2)$ is expressed analytically by*

$$r^{\lambda\nu} = \sum_{1}^{2}{}_a\, r^a \underset{a}{p}^{\lambda\nu} \tag{3.3}$$

where r^1, r^2 are variables.

PROOF. The equations (3.1) and (3.3) yield

$$r^{\lambda\nu} r_{\lambda\nu} = 0$$

$$r^{\lambda\nu} \underset{a}{p}_{\lambda\nu} = 0, \quad (a = 1, 2).$$

Hence, $r^{\lambda\nu}$ given by (3.3) are lines intersecting $\underset{a}{p}^{\lambda\nu}$ $(a = 1, 2)$. Let $s^{\lambda\nu}$ be an arbitrary line incident with the $\left\{\begin{matrix}\text{vertex}\\ \text{plane}\end{matrix}\right\}$ of the pencil P but not incident with its $\left\{\begin{matrix}\text{plane}\\ \text{vertex}\end{matrix}\right\}$. In both cases

$$s^{\lambda\nu} \underset{a}{p}_{\lambda\nu} = 0, \quad (a = 1, 2), \tag{3.4)a}$$

so that

$$s^{\lambda\nu} r_{\lambda\nu} = 0. \tag{3.4)b}$$

Thus, the lines $r^{\lambda\nu}$ are incident with the $\left\{\begin{matrix}\text{vertex}\\ \text{plane}\end{matrix}\right\}$ of P and, therefore, belong to P. Conversely, every line of P may be expressed by (3.3). In fact, if $t^{\lambda\nu}$ is a line not intersecting $\underset{a}{p}^{\lambda\nu}$,

$$t_a \overset{\text{def}}{=} t^{\lambda\nu} \underset{a}{p}_{\lambda\nu} \neq 0$$

$$(a = 1, 2),$$

then among the lines $r^{\lambda\nu}$ there is exactly one which intersects $t^{\lambda\nu}$, namely, the line for which $r^1 : r^2$ is given by

$$\sum_{1}^{2}{}_a\, r^a t_a = 0.$$

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[1] By a pencil of lines we mean the set of all lines incident with a plane and a point.

A non-degenerate quadric contains two one-parameter sets ot straight lines. Each of these sets is called a *regulus*.

THEOREM (3.3). *Let* $\underset{b}{p}^{\lambda\nu}$ $(b = 1, 2, 3)$ *be three skew lines. Put*

$$a_{cb} = a_{bc} \overset{\text{def}}{=} \underset{b}{p}_{\lambda\nu} \underset{c}{p}^{\lambda\nu}, \quad b, c = 1, 2, 3. \tag{3.5}$$

Then

$$\operatorname{Det}((a_{bc})) = 2a_{12}a_{13}a_{23} \neq 0. \tag{3.6}$$

The regulus R of lines $r^{\lambda\nu}$ *(rulings of R) determined by the lines* $\underset{a}{p}^{\lambda\nu}$ $(a = 1, 2, 3)$ *is expressed analytically by*

$$r^{\lambda\nu} = \sum_{1}^{3}{}^{a}\, r^{a} \underset{a}{p}^{\lambda\nu} \tag{3.7}$$

where the variables r^1, r^2, r^3 *satisfy the condition*

$$\sum_{1}^{3}{}^{b,c}\, r^{b} r^{c} a_{bc} = 0. \tag{3.8}$$

PROOF. The lines $\underset{a}{p}^{\lambda\nu}$ are skew, and therefore,

$$a_{12}a_{13}a_{23} \neq 0$$

$$a_{11} = a_{22} = a_{33} = 0.$$

These conditions lead easily to (3.6). From (3.7) and (3.8) we have

$$r_{\lambda\nu} r^{\lambda\nu} = 0.$$

Hence, $r^{\lambda\nu}$ are lines. Let $t^{\lambda\nu}$ be an arbitrary line and

$$t_a \overset{\text{def}}{=} t^{\lambda\nu} \underset{a}{p}_{\lambda\nu} \ (a = 1, 2, 3). \tag{3.9)a}$$

The lines $r^{\lambda\nu}$ which meet $t^{\lambda\nu}$ are given by r^1, r^2, r^3 satisfying (3.8) and

$$r^{\lambda\nu} t_{\lambda\nu} = \sum_{1}^{3}{}^{a}\, r^{a} t_{a} = 0. \tag{3.9)b}$$

From (3.6) we see that therc are only two solutions r^a $(a = 1, 2, 3)$ of (3.8) and (3.9)b, and these solutions may coincide. Hence, the lines $r^{\lambda\nu}$ generate a quadratic surface Q. Let (3.7) and

$$'r^{\lambda\nu} = \sum_{1}^{3}{}^{a}\, 'r^{a} \underset{a}{p}^{\lambda\nu} \tag{3.7)'}$$

be two distinct lines of this surface. Then we also have, besides (3.8),

(3.8)′ $$\sum_{1}^{3}{}^{b,c}\,{}'r^b\,{}'r^c a_{bc} = 0$$

and

(3.10)a $$r^{\lambda\nu}\,{}'r_{\lambda\nu} = \sum_{1}^{3}{}^{b,c}\,a_{bc}r^b\,{}'r^c.$$

Because of (3.6) it is not possible to satisfy

$$\sum_{1}^{3}{}^{b,c}\,a_{bc}r^b\,{}'r^c = 0$$

together with (3.8) and (3.8)′. Hence, we have

(3.10)b $$r^{\lambda\nu}\,{}'r_{\lambda\nu} \neq 0.$$

The lines (3.7) and (3.7)′ do not meet and, consequently, belong to the same regulus R of Q, which, according to (3.7) also contains $\underset{a}{p}^{\lambda\nu}$ ($a = 1, 2, 3$). Hence, every line (3.7) belongs to R. Conversely, every line of R is in the set (3.7). This statement follows easily from the part of the proof involving $t^{\lambda\nu}$.

NOTE. Define t_a by (3.9)a and assume that $t^{\lambda\nu}$ intersects $\underset{a}{p}^{\lambda\nu}$ ($a = 1, 2, 3$),

$$t_a = 0.$$

Then $t^{\lambda\nu}$ belongs to the second regulus R^* of Q, and we have for all rulings $r^{\lambda\nu}$ of R

$$r^{\lambda\nu}t_{\lambda\nu} = 0.$$

Hence, an arbitrary ruling $t^{\lambda\nu}$ of R^* intersects all rulings of R and vice versa.

4. Linear line complex. All lines $p^{\lambda\nu}$ intersecting a given line $q_{\lambda\nu}$ satisfy the relations

(4.1) $$q_{\lambda\nu}p^{\lambda\nu} = 0 \qquad p_{\lambda\nu}p^{\lambda\nu} = 0.$$

This conception may be generalized.

DEFINITION (4.1). *Let* $c_{\lambda\mu} = c_{[\lambda\mu]}$ *be an arbitrary tensor. The set* *of all lines* $p^{\lambda\nu}$ *satisfying the conditions*

$$(4.2) \qquad c_{\lambda\nu}p^{\lambda\nu} = 0 \qquad p_{\lambda\nu}p^{\lambda\nu} = 0$$

is called a linear line complex $c_{\lambda\mu}$, *more briefly the complex* $c_{\lambda\mu}$, *and* $p^{\lambda\nu}$ *are referred to as its rulings. If*

$$(4.3) \qquad c_{\lambda\nu}c^{\lambda\nu} \neq 0,$$

we call the complex general. If

$$(4.4) \qquad c_{\lambda\nu}c^{\lambda\nu} = 0,$$

we call the complex special and the line $c_{\lambda\mu}$ *its axis.*

NOTE. A special complex $c_{\lambda\mu}$ consists of all lines $p^{\lambda\nu}$ intersecting the axis $c_{\lambda\mu}$.

DEFINITION (4.2). *If a point* y^μ *and a plane* y_λ *are related by*

$$(4.5) \qquad a)\ \rho y_\lambda = c_{\lambda\mu}y^\mu, \quad \text{or} \quad b)\ \sigma y^\mu = c^{\lambda\mu}y_\lambda, \qquad (\rho\sigma \neq 0),$$

we call $\left\{\begin{matrix} y^\mu \\ y_\lambda \end{matrix}\right\}$ *the* $\left\{\begin{matrix} \textit{pole} \\ \textit{polar plane} \end{matrix}\right\}$ *of the* $\left\{\begin{matrix} \textit{polar plane } y_\lambda \\ \textit{pole } y^\mu \end{matrix}\right\}$ *with respect to the complex* $c_{\lambda\mu}$.

THEOREM (4.1). *If* $c_{\lambda\mu}$ *is a general complex, the polar pair* (y^μ, y_λ) *exists for every arbitrary given* $y^\mu(y_\lambda)$, *and* (4.5)b *is the inverse equation to* (4.5)a. *If* $c_{\lambda\mu}$ *is a special complex, then the pole* y^μ (*the polar plane* y_λ) *incident with the axis* $c_{\lambda\mu}$ *has no polar plane* (*has no pole*).

PROOF. A necessary and sufficient condition for the non-existence of a polar plane y_λ (of a pole y^μ) to a given pole y^μ (to a given polar plane y_λ) is $y_\lambda = 0$ ($y^\mu = 0$). Substituting this value into (4.5)a ((4.5)b) we easily obtain $\operatorname{Det}((c_{\lambda\mu})) = 0$, *i.e.*, (4.4). If this condition is satisfied, and if $y_\lambda = 0$ ($y^\mu = 0$), then the pole y^μ (the polar plane y_λ) is incident with the axis $c_{\lambda\mu}$. A straightforward computation shows that

$$(4.6) \qquad c^{\lambda\nu}c_{\lambda\mu} = \varepsilon\sqrt{c}\,\delta^\nu_\mu, \quad (c \overset{\text{def}}{=} \operatorname{Det}((c_{\lambda\mu})),\ \varepsilon \overset{\text{def}}{=} \operatorname{sgn} I^{\omega\mu\lambda\nu}c_{\omega\mu}c_{\lambda\nu}).$$

Multiplying (4.5)a by $c^{\lambda\nu}$ we obtain by virtue of (4.6)

$$\rho y_\lambda c^{\lambda\nu} = \varepsilon\sqrt{c}\,y^\nu.$$

The proof of the remaining part of the theorem follows easily from this equation.

THEOREM (4.2). *The pole y^μ and its polar plane y_λ are incident. The rulings of $c_{\lambda\mu}$ incident with y^μ (y_λ) constitute a pencil C whose plane (vertex) is y_λ (y^μ).* [1]

PROOF. The equation (4.5)a yields

$$\rho y_\lambda y^\lambda = c_{\lambda\mu} y^\lambda y^\mu = 0.$$

Hence, y^μ and y_λ are incident. Let $x^\lambda \neq y^\lambda$ be an arbitrary point in y_λ:

$$y_\lambda x^\lambda = 0.$$

Then we have from (4.5)a

$$0 = \rho y_\lambda x^\lambda = c_{\lambda\mu} y^\mu x^\lambda.$$

Hence, the line $p^{\lambda\nu} = x^{[\lambda} y^{\nu]}$ which is a generic line of C is also a ruling of the complex $c_{\lambda\mu}$.

THEOREM (4.3). *Let $s^{\lambda\nu}$ be an arbitrary line.* [2] *It determines uniquely a line $r^{\lambda\nu}$ with the following properties: every ruling of the complex $c_{\lambda\mu}$ that intersects $s^{\lambda\nu}$ intersects $r^{\lambda\nu}$; and vice versa if the complex is general. The line $r^{\lambda\nu}$ is given by*

$$r^{\lambda\nu} = c^{\lambda\nu}(2c^{\alpha\beta} s_{\alpha\beta}) - s^{\lambda\nu}(c^{\alpha\beta} c_{\alpha\beta}). \tag{4.7}$$

If the complex $c_{\lambda\mu}$ is general, then the lines $r^{\lambda\nu}$ and $s^{\lambda\nu}$ coincide if and only if one of them is a ruling of $c_{\lambda\mu}$. If this condition is not satisfied, then $r^{\lambda\nu}$ and $s^{\lambda\nu}$ do not intersect. If the complex is special, then $r^{\lambda\nu}$ coincides with the axis.

PROOF. A necessary and sufficient condition for a ruling $p^{\lambda\nu}$ of $c_{\lambda\mu}$ to intersect $s^{\lambda\nu}$ is that $p^{\lambda\nu}$ belong to every one of the complexes

$$r_{\lambda\mu} \overset{\text{def}}{=} \alpha c_{\lambda\mu} + \beta s_{\lambda\mu} \tag{4.8}$$

where α, β are variable parameters. There are two lines in the set

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[1] If $c_{\mu\lambda}$ is special, we tacitly assume that y^μ (y_λ) is not incident with the axis $c_{\lambda\mu}$.

[2] If the complex under consideration is special, we impose the further condition that $s^{\lambda\nu}$ does not belong to the complex (and, therefore, that $s^{\lambda\nu}$ is not its axis.).

(4.8) defined by parameters α, β which satisfy the condition

$$r^{\lambda\nu}r_{\lambda\nu} = \alpha[\alpha c^{\lambda\nu}c_{\lambda\nu} + 2\beta c^{\lambda\nu}s_{\lambda\nu}] = 0.$$

The value $\alpha = 0$ leads to $s_{\lambda\mu}$ while

$$\alpha : \beta = 2c^{\alpha\beta}s_{\alpha\beta} : - c^{\alpha\beta}c_{\alpha\beta}$$

leads to (4.7), and this equation yields

$$r^{\lambda\nu}s_{\lambda\nu} = 2(c^{\alpha\beta}s_{\alpha\beta})^2. \tag{4.9}$$

The remaining part of the Theorem follows easily from (4.7) and (4.9).

DEFINITION (4.3). *The lines* $r^{\lambda\nu}$ *and* $s^{\lambda\nu}$ *of Theorem* (4.3) *are said to be conjugate polars with respect to the complex* $c_{\lambda\mu}$, *or briefly, polar.*

5. Pencil of complexes. In subsequent considerations we shall start with two distinct real general complexes $\underset{a}{c}_{\lambda\mu}$[1] $(a = 1, 2)$ which give rise to the following scalars

$$a_{bc} = a_{cb} \overset{\text{def}}{=} \underset{c}{c}_{\lambda\mu}\underset{b}{c}^{\lambda\mu} \quad (b, c = 1, 2) \tag{5.1)a}$$

$$a^2 \overset{\text{def}}{=} (a_{12})^2 - a_{11}a_{22}, \tag{5.1)b}$$

$$a \overset{\text{def}}{=} \sqrt{a^2}. \tag{5.1)c}$$

DEFINITION (5.1). *If* $a \neq 0$, *then the one-parameter set of complexes*

$$c_{\lambda\mu} \overset{\text{def}}{=} \sum_{1}^{2}{}^{a}\, c^{a}\underset{a}{c}_{\lambda\mu} \tag{5.2}$$

with the variable $c^1 : c^2$ *will be referred to as a pencil of complexes.*[2]

THEOREM (5.1). *The sign*

$$\alpha \overset{\text{def}}{=} \operatorname{sgn} a^2 \tag{5.3}$$

does not depend on the particular choice of the real complexes $\underset{1}{c}_{\lambda\mu}$ *and* $\underset{2}{c}_{\lambda\mu}$ *in the pencil* (5.2).

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[1] By a real complex $c_{\lambda\mu}$ we mean a complex whose components are all real.

[2] For the excluded case $a = 0$ see Hlavatý, V., (11).

PROOF. Let

(5.4)a $${}'\underset{a}{c}_{\lambda\mu} = \sum_{1}^{2}{}^{b}\, u_a{}^b \underset{b}{c}_{\lambda\mu}, \quad (a = 1, 2)$$

be two distinct real complexes of the pencil so that

(5.4)b $$u \overset{\text{def}}{=} \operatorname{Det}((u_a{}^b)) \neq 0.$$

A simple inspection shows that

(5.5)a $${}'a_{bc} \overset{\text{def}}{=} {}'\underset{b}{c}_{\lambda\mu}\, {}'\underset{c}{c}^{\lambda\mu} = \sum_{1}^{2}{}^{i,j}\, u_b{}^i u_c{}^j a_{ij},$$

and consequently,

(5.5)b $${}'a^2 \overset{\text{def}}{=} ({}'a_{12})^2 - {}'a_{11}\,{}'a_{22} = u^2 a^2.$$

The complexes ${}'\underset{a}{c}_{\lambda\mu}$, $\underset{b}{c}_{\lambda\mu}$ being real, the numbers $u_a{}^b$ involved in (5.4)a must also be real, and therefore, by virtue of (5.4)b, $u^2 > 0$. Theorem (5.1) follows easily from (5.5)b.

THEOREM (5.2). *There exists a unique pair of lines* $\underset{1}{f}^{\lambda\nu}$, $\underset{2}{f}^{\lambda\nu}$ *with the following properties*:

1) *They do not meet.*

2) *They are* $\left\{\begin{matrix}\textit{real} \\ \textit{complex conjugates}\end{matrix}\right\}$ *for* $\begin{matrix}\alpha > 0 \\ \alpha < 0\end{matrix}$.

3) *They are conjugate polars with respect to all general complexes of the pencil* (5.2).

4) *They are axes of the only two special complexes of the pencil* (5.2).

5) *They do not depend on the particular choice of the complexes* $\underset{a}{c}_{\lambda\mu}$ *in the pencil* (5.2). [1]

6) *They are given by*

(5.6)a $$\underset{1}{f}^{\lambda\nu} = \underset{1}{c}^{\lambda\nu}(a - a_{12}) + \underset{2}{c}^{\lambda\nu} a_{11}$$

(5.6)b $$\underset{2}{f}^{\lambda\nu} = \underset{1}{c}^{\lambda\nu}(-\,a - a_{12}) + \underset{2}{c}^{\lambda\nu} a_{11}.$$

PROOF. Special complexes in the pencil (5.2) are defined by the parameters c^a $(a = 1, 2)$ satisfying the condition

(5.7)a $$c_{\lambda\mu} c^{\lambda\mu} = \sum_{1}^{2}{}^{b,c}\, c^b c^c a_{bc} = 0,$$

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[1] At this point we permit the complexes $\underset{a}{c}_{\lambda\mu}$ to be other than real.

i.e.,

$$c^1 : c^2 = (\pm\, a - a_{12}) : a_{11}. \qquad \text{(5.7)b}$$

Substituting these values into (5.2) we obtain (5.6) which prove statements 4) and 6). It is seen at once from (5.6) that the lines $\underset{a}{f}^{\lambda\nu}$ $(a = 1, 2)$ are real for $\alpha > 0$ and complex conjugates for $\alpha < 0$. Moreover, by (5.1)a

$$\underset{1}{f}^{\lambda\nu}\underset{2}{f}_{\lambda\nu} = -\,2a_{11}a^2 \neq 0. \qquad \text{(5.8)}$$

This equation proves statement 1). If $p^{\lambda\nu}$ is a common ruling of $\underset{1}{c}_{\lambda\mu}$ and $\underset{2}{c}_{\lambda\mu}$, then it is a ruling of every complex of (5.2) and, therefore, also of every special complex $\underset{1}{f}^{\lambda\nu}$, $\underset{2}{f}^{\lambda\nu}$ of this pencil. Therefore,

$$\underset{1}{f}_{\lambda\nu}p^{\lambda\nu} = \underset{2}{f}_{\lambda\nu}p^{\lambda\nu} = 0,$$

and this equation proves statement 3). If we refer the pencil (5.2) to the complexes (5.4)a [1] and write

$$c_{\lambda\mu} = \sum_{1}^{2}{}_a\, {}'c^a\, {}'\underset{a}{c}_{\lambda\mu},$$

then

$$\sum_{1}^{2}{}_a\, {}'c^a u_a{}^b = c^b, \qquad (b = 1, 2),$$

so that according to (5.5)a

$$\sum_{1}^{2}{}_{b,c}\, a_{bc}c^b c^c = \sum_{1}^{2}{}_{b,c}\, {}'a_{bc}{}'c^b{}'c^c. \qquad \text{(5.9)}$$

The proof of statement 5) follows easily from (5.9), (5.7)a and

$$f^{\lambda\nu} = \sum_{1}^{2}{}_a\, {}'c^a\, {}'\underset{a}{c}^{\lambda\nu} = \sum_{1}^{2}{}_a\, c^a \underset{a}{c}^{\lambda\nu}.$$

NOTE. The parameters c^a in (5.2) may be looked upon as projective homogeneous coordinates of $c_{\lambda\mu}$ with respect to $\underset{a}{c}_{\lambda\mu}$. We stress this fact by writing $c_{\lambda\mu}(c^1, c^2)$ instead of (5.2). Then we may represent $\underset{1}{c}_{\lambda\mu}, \underset{2}{c}_{\lambda\mu}, \underset{1}{f}_{\lambda\mu}, \underset{2}{f}_{\lambda\mu}$ by

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[1] For present considerations, the complexes ${}'\underset{a}{c}_{\lambda\mu}$ need not necessarily be real.

$$\begin{aligned} &\underset{1}{c}_{\lambda\mu}(1, 0) \\ &\underset{2}{c}_{\lambda\mu}(0, 1) \\ &\underset{1}{f}_{\lambda\mu}(a - a_{12}, a_{11}) \\ &\underset{2}{f}_{\lambda\mu}(-a - a_{12}, a_{11}). \end{aligned} \tag{5.10}$$

Denoting by $(\underset{1}{c}, \underset{2}{c}, \underset{1}{f}, \underset{2}{f})$ the cross-ratio of the complexes (5.10) we easily obtain

$$(\underset{1}{c}, \underset{2}{c}, \underset{1}{f}, \underset{2}{f}) = \frac{a_{12} + a}{a_{12} - a}. \tag{5.11}$$

6. Continuation: Projective angle of complexes. In this section we again use the complexes $\underset{a}{c}_{\lambda\mu}$ $(a = 1, 2)$ of the previous section and the scalars (5.1) and (5.3).

DEFINITION (6.1). *If* $a \neq 0$, *then the scalar*

$$\varphi(\underset{1}{c}, \underset{2}{c}) \overset{\text{def}}{=} \frac{\sqrt{\alpha}}{2} \ln \frac{a_{12} + a}{a_{12} - a}. \tag{6.1}$$

is called the projective angle of the complexes $\underset{1}{c}_{\lambda\mu}, \underset{2}{c}_{\lambda\mu}$. *If, in particular,*

$$\varphi(\underset{1}{c}, \underset{2}{c}) = \pm \frac{\pi}{2} \sqrt{-\alpha}, \tag{6.2}$$

then $\underset{1}{c}_{\lambda\mu}$ *and* $\underset{2}{c}_{\lambda\mu}$ *are said to be projectively orthogonal.*

THEOREM (6.1). *A necessary and sufficient condition for the complexes* $\underset{1}{c}_{\lambda\mu}$ *and* $\underset{2}{c}_{\lambda\mu}$ *to be projectively orthogonal is*

$$a_{12} = 0. \tag{6.3}$$

PROOF. Substituting from (6.2) into (6.1) one obtains

$$\frac{a_{12} + a}{a_{12} - a} = e^{\pm \pi i} = -1,$$

and this equation is equivalent to (6.3).

In order to find a geometrical interpretation of the projective angle $\varphi(\underset{1}{c}, \underset{2}{c})$ let us first observe that if y^μ is an arbitrary real point not incident with $\underset{a}{f}^{\lambda\nu}$ $(a = 1, 2)$, its polar plane $c_{\lambda\mu}y^\mu$ with respect to the generic complex $c_{\lambda\mu}$ of the pencil (5.2) always exists and is contained in the pencil

$$(6.4) \qquad c_{\lambda\mu}y^\mu = \overset{2}{\underset{1}{\sum}}{}^{a}\, c^a \underset{a}{c}_{\lambda\mu}y^\mu,$$

defined by the polar planes $\underset{a}{c}_{\lambda\mu}y^\mu$ of y^μ with respect to the complexes $\underset{a}{c}_{\lambda\mu}$. In particular, the polar planes

$$(6.5) \qquad \begin{aligned} \underset{1}{f}_\lambda &\overset{\text{def}}{=} \underset{1}{f}_{\lambda\mu}y^\mu = (a - a_{12})\underset{1}{c}_{\lambda\mu}y^\mu + a_{11}\underset{2}{c}_{\lambda\mu}y^\mu \\ \underset{2}{f}_\lambda &\overset{\text{def}}{=} \underset{2}{f}_{\lambda\mu}y^\mu = (-\,a - a_{12})\underset{1}{c}_{\lambda\mu}y^\mu + a_{11}\underset{2}{c}_{\lambda\mu}y^\mu \end{aligned}$$

belong to the pencil (6.4), and according to statement 5) of Theorem (5.2), they do not depend on the choice of the complexes $\underset{a}{c}_{\lambda\mu}$ in the pencil (5.2). They are intrinsically connected with the pencil (6.4), and therefore, this pencil may be looked upon *as a non-Euclidean pencil with absolute planes* (6.5). In such a pencil the non-Euclidean angle $\psi[\underset{1}{c}, \underset{2}{c}]$ of the planes $\underset{a}{c}_\lambda = \underset{a}{c}_{\lambda\mu}y^\mu$ $(a = 1, 2)$ is defined by the known Laguerre formula

$$(6.6) \qquad \psi[\underset{1}{c}, \underset{2}{c}] = \frac{\sqrt{\gamma}}{2}\ln\,[\underset{1}{c}, \underset{2}{c}, \underset{1}{f}, \underset{2}{f}]$$

where $[\underset{1}{c}, \underset{2}{c}, \underset{1}{f}, \underset{2}{f}]$ is the cross-ratio of the four planes $\underset{a}{c}_\lambda$ and $\underset{a}{f}_\lambda$ $(a = 1, 2)$, and $\gamma = +\,1$ $(-\,1)$ if $\underset{a}{f}_\lambda$ are real (complex conjugates). We shall use this non-Euclidean angle in

THEOREM (6.2). *An arbitrary real point y^μ not incident with the lines* (5.6) *gives rise to a non-Euclidean pencil* (6.4) *of polar planes with the absolute planes* (6.5). *The non-Euclidean angle $\psi[\underset{1}{c}, \underset{2}{c}]$ of the polar planes*

$$(6.7) \qquad \underset{a}{c}_\lambda = \underset{a}{c}_{\lambda\mu}y^\mu, \quad (a = 1, 2)$$

of the pencil (6.4) *is equal to the projective angle* $\varphi(\underset{1}{c}, \underset{2}{c})$ *of the complexes* $\underset{a}{c}_{\lambda\mu}$ $(a = 1, 2)$

$$\varphi(\underset{1}{c}, \underset{2}{c}) = \psi[\underset{1}{c}, \underset{2}{c}]. \tag{6.8}$$

The non-Euclidean angle $\psi(\underset{1}{c}, \underset{2}{c})$ *does not depend on the choice of the point* y^μ.

PROOF. The content of the first part of the theorem is merely a restatement of results just obtained.

The parameters c^a $(a = 1, 2)$ involved in (6.4) may be looked upon as homogeneous projective coordinates of the plane $c_{\lambda\mu}\, y^\mu$ with respect to the planes (6.7). If we write $c_{\lambda\mu} y^\mu(c^1, c^2)$ instead of (6.4), we have

$$\begin{aligned} &\underset{1}{c}_\lambda(1, 0)\\ &\underset{2}{c}_\lambda(0, 1)\\ &\underset{1}{f}_\lambda(a - a_{12}, a_{11})\\ &\underset{2}{f}_\lambda(-\, a - a_{12}, a_{11}), \end{aligned} \tag{6.9}$$

and therefore, according to (5.10) and (5.11)

$$[\underset{1}{c}, \underset{2}{c}, \underset{1}{f}, \underset{2}{f}] = (\underset{1}{c}, \underset{2}{c}, \underset{1}{f}, \underset{2}{f}) = \frac{a_{12} + a}{a_{12} - a}. \tag{6.10)a$$

If $\alpha = +\,1$ $(\alpha = -\,1)$, then according to Theorem (5.2) the lines (5.6) are real (complex conjugates). Hence the planes (6.5) are real (complex conjugates), and therefore, γ involved in (6.6) is equal to 1 (equal to $-\,1$). Thus,

$$\alpha = \gamma. \tag{6.10)b$$

The equation (6.8) follows from (6.10), (6.6) and (6.1). The left-hand term in (6.8) does not depend on the choice of the point y^μ, and, therefore, the right-hand term in (6.8) is also independent of this choice.

REMARK. Notice that for different points y^μ we might get different non-Euclidean pencils (6.4), but $\psi[\underset{1}{c}, \underset{2}{c}]$ remains the same for all such pencils.

7. Linear line congruence.

DEFINITION (7.1). *The set of rulings common to all complexes of the pencil* (5.2) *is called a linear line congruence, briefly, a congruence. The lines* (5.6) *are called the axes of the congruence.*

THEOREM (7.1)a. *The congruence defined in the previous definition may be expressed analytically as follows*:

$$\underset{a}{c}_{\lambda\mu} p^{\lambda\mu} = 0, \quad p_{\lambda\mu} p^{\lambda\mu} = 0, \quad (a = 1 ,2). \tag{7.1)a}$$

This set of three equations is equivalent to

$$\underset{a}{f}_{\lambda\mu} p^{\lambda\mu} = 0, \quad p_{\lambda\mu} p^{\lambda\mu} = 0, \quad (a = 1, 2). \tag{7.1)b}$$

The proof follows the pattern of the proof of statement 3) of Theorem (5.2).

THEOREM (7.1)b. *The congruence* (7.1)a *consists of all lines intersecting the axes* $\underset{a}{f}^{\lambda\nu}$.

The proof follows at once from the equivalence of (7.1)a and (7.1)b and from Theorem (3.1).

THEOREM (7.2). *Let* y^μ (y_λ) *be an arbitrary point* (*plane*) *not incident with the axes* $\underset{a}{f}^{\lambda\nu}$ $(a = 1, 2)$. *Then there is only one ruling* $p_{\lambda\mu}$ *of the congruence incident with* y^μ *and only one ruling* $q^{\lambda\nu}$ *of the congruence incident with* y_λ, *namely,*

$$p_{\lambda\mu} = y^\alpha y^\beta \underset{1}{f}_{\alpha[\lambda} \underset{2}{f}_{\mu]\beta} \tag{7.2)a}$$

$$q^{\lambda\mu} = y_\alpha y_\beta \underset{1}{f}^{\alpha[\lambda} \underset{2}{f}^{\mu]\beta}. \tag{7.2)b}$$

PROOF. If y^μ is not incident with $\underset{a}{f}^{\lambda\nu}$ $(a = 1, 2)$, there is only one line $p_{\lambda\mu}$ through y^μ and intersecting these axes, namely, the intersection line of the planes (6.5).[1] Therefore, $p_{\lambda\mu}$ is defined by (7.2)a. The remaining part of the Theorem may be proved in a similar way by means of intersection points $\underset{a}{f}^{\lambda\nu} y_\nu$ $(a = 1, 2)$ of y_λ and $\underset{a}{f}^{\lambda\nu}$ $(a = 1, 2)$.

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[1] This line is also the intersection line of any arbitrary pair of distinct planes of the pencil (6.4).

THEOREM (7.3). *Let* $\underset{a}{y^{\mu}}$ ($\overset{a}{y_{\lambda}}$) *be a point* (*plane*) *incident with* $\underset{a}{f^{\lambda\nu}}$ ($a = 1, 2$). *Then the rulings of the congruence incident with* $\underset{a}{y^{\mu}}$ ($\overset{a}{y_{\lambda}}$) *constitute a pencil whose vertex* (*plane*) *is* $\underset{a}{y^{\nu}}$ ($\overset{a}{y_{\lambda}}$) *and whose plane* (*vertex*) *is* $\underset{a}{y^{\lambda}}\underset{b}{f_{\lambda\mu}}$ ($\overset{a}{y_{\lambda}}\underset{b}{f^{\lambda\nu}}$), $a, b = 1, 2$; $a \neq b$.

PROOF. Every ruling of the congruence intersects both axes $\underset{1}{f^{\lambda\nu}}$, $\underset{2}{f^{\lambda\nu}}$. Hence, if $\underset{a}{y^{\lambda}}$ is incident with $\underset{a}{f^{\lambda\nu}}$, then the rulings incident with $\underset{a}{y^{\lambda}}$ constitute a pencil whose plane is defined by $\underset{a}{y^{\lambda}}$ and $\underset{b}{f_{\lambda\mu}}$ ($a, b = 1, 2$; $a \neq b$). This plane is obviously $\underset{a}{y^{\lambda}}\underset{b}{f_{\lambda\mu}}$.[1] The remaining statement may be proved in a similar manner.

DEFINITION (7.2). *Let* $p^{\lambda\nu}$ *be a ruling of the congruence* (7.1). *The point* (*plane*) *defined by* $p^{\lambda\nu}$ *and* $\underset{a}{f^{\lambda\nu}}$ ($a = 1$ *or* $a = 2$) *is called the a-th focal point* (*focal plane*) *of* $p^{\lambda\nu}$.

NOTE. Let $p^{\lambda\nu}_{\circ}$, the intersection point of $p^{\lambda\nu}$ and the plane (2.11) be not incident with $\underset{1}{f^{\lambda\nu}}$ or $\underset{2}{f^{\lambda\nu}}$. The polar plane of $p^{\lambda\nu}_{\circ}$ with respect to the special complex $\underset{a}{f_{\lambda\mu}}$ ($a = 1$ or $a = 2$) has the coordinates $p^{\lambda\nu}_{\circ}\underset{a}{f_{\lambda\mu}}$. These two planes (for $a = 1, 2$) are the focal planes of $p^{\lambda\nu}$. Let $\underset{\circ}{p_{\lambda\mu}}$ be the plane through $p_{\lambda\mu}$ and the coordinate tetrahedron vertex $\underset{\circ}{x_{\mu}} = 0$ not incident with $\underset{1}{f^{\lambda\nu}}$ or $\underset{2}{f^{\lambda\nu}}$. The pole of $\underset{\circ}{p_{\lambda\mu}}$ with respect to the special complex $\underset{a}{f^{\lambda\nu}}$ ($a = 1$ or $a = 2$) has the coordinates $\underset{\circ}{p_{\lambda\mu}}\underset{a}{f^{\lambda\nu}}$. These two points (for $a = 1, 2$) are focal points of $p_{\lambda\mu}$. Focal points and focal planes are very important conceptions in the theory of biaxial projectivities connected with the congruence (7.1).

8. Biaxial projectivity.

THEOREM (8.1). *For every congruence* (7.1) *there exists a one-parameter set of tensors* $P_{\lambda}{}^{\nu}$ *with the following properties*:

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[1] This plane is the polar plane of $\underset{a}{y^{\lambda}}$ with respect to all complexes of the pencil (5.2) with the exception of the special complex $\underset{a}{f^{\lambda\nu}}$ which does not have the polar plane of $\underset{a}{y^{\lambda}}$ (*Cf.* Theorem (4.1)).

a) *The tensor* $P_\lambda{}^\nu$ *is not a degenerate one, i.e.,*

(8.1) $$\mathrm{Det}((P_\lambda{}^\nu)) \neq 0.$$

b) *It satisfies the condition*

(8.2)a $$z^\lambda z^{[\mu} P_\lambda{}^{\nu]} \underset{a}{f}_{\mu\nu} = 0, \quad (a = 1, 2),$$

for any arbitrary point z^ν.

c) $P_\lambda{}^\nu$ *is not proportional to* δ_λ^ν

(8.2)b $$P_\lambda{}^\nu \neq \rho\delta_\lambda^\nu, \ (\rho \neq 0).$$

PROOF. Assume first that $\underset{a}{f}_{\lambda\mu}$ are real, and choose the coordinate system in such a way that

(8.3)a $$\underset{1}{f}_{\mathrm{I\ II}}\ \underset{2}{f}_{\mathrm{III\ IV}} \neq 0$$

8.3)b $$\text{the remaining } \underset{a}{f}_{\lambda\mu} = 0.$$

An easy inspection shows that under these circumstances the conditions (8.1) and (8.2) yield

(8.4)a $$P_{\mathrm{I}}{}^{\mathrm{I}} = P_{\mathrm{II}}{}^{\mathrm{II}},\ P_{\mathrm{III}}{}^{\mathrm{III}} = P_{\mathrm{IV}}{}^{\mathrm{IV}},\ P_{\mathrm{I}}{}^{\mathrm{I}} \neq P_{\mathrm{III}}{}^{\mathrm{III}}$$

(8.4)b $$\text{the remaining } P_\lambda{}^\nu = 0.$$

The components $P_\lambda{}^\nu$ being homogeneous, the equations (8.4) define a one-parameter set of tensors $P_\lambda{}^\nu$. If $\underset{a}{f}_{\lambda\mu}$ are complex conjugates, then the proof follows the same pattern though it is a little bit more complicated.

THEOREM (8.2). *Every tensor* $P_\lambda{}^\nu$ *satisfying* (8.1) *and* (8.2) *gives rise to a biaxial projectivity*

(8.5) $$'z^\nu = P_\lambda{}^\nu z^\lambda,$$

i.e., to a non-degenerate projectivity in which the lines $'z^{[\nu} z^{\lambda]}$ *intersect both axes* $\underset{1}{f}^{\lambda\nu}$ *and* $\underset{2}{f}^{\lambda\nu}$. *These two axes constitute the locus of double points of* (8.5).

PROOF. Put

(8.6)a $$p^{\lambda\nu} = {}'z^{[\lambda} z^{\nu]}.$$

Then we have according to (8.5)

(8.6)b $$p^{\lambda\nu} = z^\alpha P_\alpha{}^{[\lambda} z^{\nu]}$$

and by virtue of (8.2)a

(8.7) $$p^{\lambda\nu} \underset{a}{f}_{\lambda\nu} = z^\alpha P_\alpha{}^{[\lambda} z^{\nu]} \underset{a}{f}_{\lambda\nu} = 0.$$

Hence if $\rho' z^\nu \neq z^\nu$, the line (8.6) intersects $\underset{a}{f}_{\lambda\mu}$, $(a = 1,2)$.

Assume that the axes $\underset{a}{f}_{\lambda\mu}$ are real. Using (8.3) and (8.4) we see at once that all points of these axes and only these points are double points of the projectivity (8.5). If the axes $\underset{a}{f}_{\lambda\mu}$, $(a = 1, 2)$ are not real then the proof follows a similar pattern though somewhat more involved.

NOTE. Denote by $\underset{a}{f}^\nu$ the focal points on (8.6). They satisfy the relation

$$(8.8) \qquad \underset{a}{f}^\lambda \underset{a}{f}_{\lambda\mu} = 0, \quad (a = 1, 2).$$

They are the only double points of the projectivity (8.5) incident with the line (8.6). The points $'z^\nu, z^\nu, \underset{1}{f}^\nu, \underset{2}{f}^\nu$ being incident with the line (8.6), we may write

$$(8.9)\text{a} \qquad 'z^\nu = \sum_1^2{}_a\, '\zeta^a \underset{a}{f}^\nu$$

$$z^\nu = \sum_1^2{}_a\, \zeta^a \underset{a}{f}^\nu.$$

Moreover, the double points $\underset{a}{f}^\nu$ satisfy the conditions

$$(8.9)\text{b} \qquad \underset{a}{\rho}\, \underset{a}{f}^\nu = P_\lambda{}^\nu \underset{a}{f}^\lambda, \quad (a = 1, 2),$$

where $\underset{1}{\rho}, \underset{2}{\rho} \neq \underset{1}{\rho}$ are well-defined functions of $P_\lambda{}^\nu$,[1] both different from zero and independent of $\underset{a}{f}^\nu$.

THEOREM (8.3). *Every line* (8.6)a *is a double line of the biaxial projectivity* (8.5). *This projectivity induces on each such double line the projectivity*

$$(8.10) \qquad '\zeta^a = \underset{a}{\rho}\, \zeta^a, \quad (a = 1, 2).$$

---

[1] $\underset{1}{\rho}, \underset{2}{\rho}$ are the only roots (each counted twice) of the equation

$$\mathrm{Det}((\rho\delta_\lambda^\nu - P_\lambda^\nu)) = 0.$$

If (8.4) hold, then this equation is equivalent to

$$(\rho - P_{\mathrm{I}}{}^{\mathrm{I}})^2\, (\rho - P_{\mathrm{III}}{}^{\mathrm{III}})^2 = 0.$$

The cross-ratio

(8.11)a $$P = ({}'z^\nu, z^\nu, \underset{1}{f^\nu}, \underset{2}{f^\nu})$$

(the characteristic of the biaxial projectivity) is given by

(8.11)b $$P = \underset{2}{\rho}/\underset{1}{\rho},$$

and does not depend on the choice of the point z^ν *in space (provided* z^ν *is not incident with the axes* $\underset{a}{f^{\lambda\nu}}$ $(a = 1, 2)$*).*

PROOF. The double points $\underset{a}{f^\nu}$ are on the line (8.6)a,[1] and, therefore, $p^{\lambda\nu}$ itself must be a double line of the biaxial projectivity. Substituting into (8.5) from (8.9)a and (8.9)b we obtain

$$\sum_{1}^{2}{}^{a}\,({}'\zeta^a - \underset{a}{\rho}\,\zeta^a)\underset{a}{f^\nu} = 0,$$

and this equation yields (8.10) because the points $\underset{a}{f^\nu}$ $(a = 1, 2)$ are linearly independent. The equation (8.11)b follows easily from (8.9) and (8.10). The scalars $\underset{1}{\rho}$, $\underset{2}{\rho}$ are independent of z^ν, and so is P.

NOTE. Let z^ν be an arbitrary point not incident with the axes. In applying the projectivity (8.5) twice we obtain a point ${}'({}'z^\nu)$ which is not necessarily identical with z^ν. If for at least one point z^ν (not incident with the axes) the point ${}'({}'z^\nu)$ is identical with z^ν, then this property holds for every point, and the biaxial projectivity is called a *biaxial involution.*

THEOREM (8.4). *A necessary and sufficient condition that the biaxial projectivity* (8.5) *reduce to a biaxial involution is*

(8.12) $$P_\lambda{}^\nu P_\mu{}^\lambda = \omega\delta^\nu_\mu, \quad (\omega \neq 0).$$

PROOF. Applying the projectivity (8.5) twice to z^ν we obtain

(8.13) $${}'({}'z^\nu) = P_\lambda{}^\nu P_\mu{}^\lambda z^\mu.$$

Our Theorem is a consequence of (8.13).

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[1] This line and its double points depend on the choice of the point z^ν in the space. This point is assumed not to be incident with the axes $\underset{a}{f^{\lambda\nu}}$ $(a = 1, 2)$.

THEOREM (8.5). *The characteristic P of a biaxial involution is*

$$P = -1. \tag{8.14}$$

PROOF. Applying the induced projectivity (8.10) twice we obtain

$$'('\zeta^a) = (\underset{a}{\rho})^2\zeta^a.$$

Hence, for a biaxial involution we must have

$$\underset{1}{\rho}^2 = \underset{2}{\rho}^2. \tag{8.15)a}$$

From (8.2)b we know that a biaxial involution is not an identity. Its only double points are the points of the axes. Hence, the induced projectivity cannot be an identity. Thus, the only possible solution of (8.15) is

$$\underset{1}{\rho} = -\underset{2}{\rho}. \tag{8.15)b}$$

Substituting this value into (8.11)b one obtains (8.14).

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